Optimal topology of trusses or perforated deep beams with rotational restraints at both ends

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Abstract This note discusses the least-weight layout of a truss having support conditions similar to those for a clamped deep beam. Numerical solutions are presented and confirmed by analytical considerations based on Michell's optimality criteria.

1 Introduction

The theory of least-weight trusses was pioneered around the turn of the century by the Australian scientist, A.G.M. Michell (1904), who also determined the optimal layout for trusses with two simple supports and a point load (Fig. 1a). Hemp (1974) derived the least-weight truss design for distributed load between simple supports (Fig. 1b). It was shown later by his associate, Chan (1975) that the above layout is optimal only for a certain range of non-uniformly distributed loads. The optimal Michell layout for a cantilever truss is trivial, consisting of two bars at ±45° to the horizontal, if the height h of the vertical support is sufficiently large in comparison to the distance L of the force P from the support (i.e. if $L \leq h/2$ in Fig. 1c). If the above inequality is violated ($L > h/2$ in Fig. 1c), then the optimal truss layout becomes much more complicated (Chan 1960; reviewed in a book by Hemp 1973, pp. 97-101). This note is concerned with the optimal layout of trusses with short vertical supports at both ends (with $L > h/2$, Fig. 1d).

2 Assumed topology and numerical solutions

Before any numerical or analytical results were obtained, it was hypothesized that the optimal layout consists partly of Michell's (1904) solution for simple supports ("suspended" part of the truss in Fig. 2a) and partly of Chan's (1960) solution for cantilever trusses (the two cantilevers in Fig. 2a). The above solution consists of some circular "fans" (e.g. $ACD$, $BCE$, $IGJ$, $IHK$), "concentrated" members (e.g. $AD$, $BE$, $DF$, $EF$, $FG$, $FH$, $GJ$, $HK$) and Hencky nets consisting of curved members ($CDFE$).
the topology in Fig. 2b is very similar to the expected one (Fig. 2a), except that at this resolution (2592 square elements if we ignore symmetry) the thin members in some circular fans (e.g. ACD) do not appear in the numerical solution. This is understandable because the latter are very light compared to the "concentrated" members. Using a discretized optimality criteria method for truss optimization (Zhou and Rozvany 1991) with 13994 truss elements and 21 x 15 grid points, Gerdes obtained the solution in Fig. 2c, which again confirms the topology in Fig. 2a, with small differences due to discretization. It was observed in both numerical results (Figs. 2b and c) that the thick members in the cantilever parts enclose about 37 degrees with the horizontal.

3 Analytical solution

Michell’s (1904) optimality criteria for least weight trusses can be stated as

\[ \varepsilon_i = k \text{sgn} N_i \quad \text{for} \quad N_i \neq 0, \]

\[ |\varepsilon_i| \leq k \quad \text{for} \quad N_i = 0, \]

where \( \varepsilon_i \) are principal strains, \( N_i \) are bar forces in the principal directions, \( k \) is a given constant and \( \text{sgn} \) is the usual sign function (\( \text{sgn} N_i = 1 \) for \( N_i > 0 \) and \( \text{sgn} N_i = -1 \) for \( N_i < 0 \)). This means that all non-vanishing bars must be in the principal directions with a strain of constant absolute value (\( |\varepsilon_i| = k \)).

Figure 3 shows the principal strain directions satisfying the optimality condition (1) for all regions of the solution in Fig. 2a which are associated with non-vanishing members. Details of the displacement fields in the most important regions, as derived by Lewiński, are given below.

3.1 Region IGL

For this region, Hemp (1973, p. 81) derived the following displacements in polar coordinates (\( r, \theta \)) relative to the point I (Fig. 3):

\[ u_r = kr, \quad u_\theta = -2kr\theta. \]

where \( u_y \) is the vertical (upward) displacement at point I.

3.2 Region FGIH

\[ u_x = k(1 - \pi/2) y, \quad u_y = k(1 + \pi/2)x + c. \]

It can be checked easily that the above displacement fields give the principal strains \( \varepsilon = k \) and \( \varepsilon = -k \) in the directions indicated in Fig. 3. Moreover, continuity conditions are satisfied by the equations (4) and (6) along the region boundary GI.

3.3 Region CDFE

The treatment of this region, together with the regions ACD, BCE and ABC, was discussed by Hemp (1973, pp. 97-99) who used Chan’s (1960) results. The lines of principal strains are parameterized in curvilinear coordinates \((\alpha, \beta)\), using the Lamé coefficients

\[ A(\alpha, \beta) = 2[I_0(\gamma) + \sqrt{\beta}I_1(\gamma)], \]

\[ B(\alpha, \beta) = A(\beta, \alpha), \]

where \( \gamma = 2\sqrt{\beta} \), \( I_0 \) and \( I_1 \) are modified Bessel functions and the distance \( \varphi \) is shown in Fig. 3. By inverting Hemp’s (1973) relations (4.12), we have

\[ \frac{\partial \alpha}{\partial \xi} = \frac{\cos \varphi}{A}, \quad \frac{\partial \alpha}{\partial \eta} = \frac{\sin \varphi}{A}, \quad \frac{\partial \beta}{\partial \xi} = -\frac{\sin \varphi}{B}, \quad \frac{\partial \beta}{\partial \eta} = \frac{\cos \varphi}{B}, \]

where \( \varphi \) is the angle enclosed by the tangent of the \( \alpha \)-lines and the \( \xi \)-axis, with the additional relation \( \varphi = -\alpha + \beta \). The displacements in the region CDFE are given by (Hemp, 1973, Eq. 4.122)

\[ u_\alpha(\alpha, \beta) = k\beta[1 + 2\alpha]I_0(\gamma) + \gamma I_1(\gamma), \]

\[ u_\beta(\alpha, \beta) = -u_\alpha(\beta, \alpha). \]

The vertical displacement \( u_y \) can be expressed in terms of \((u_\alpha, u_\beta)\) by simple coordinate transformations

\[ u_y = \frac{-\sqrt{2}}{2} \left[ \cos \varphi (u_\alpha - u_\beta) - \sin \varphi (u_\alpha + u_\beta) \right]. \]

Substituting (9) into (10), we express \( u_y \) in terms of \( \alpha \) and \( \beta \) as follows:

\[ u_y = -\frac{\sqrt{2}}{2} k\beta[w_1 + w_2], \]

\[ w_1 = \cos \varphi (1 + \alpha + \beta)I_0(\gamma) + \gamma I_1(\gamma), \]

\[ w_2 = (\beta - \alpha)\sin \varphi I_0(\gamma). \]

We denote the value of the curvilinear coordinates \( \alpha \) and \( \beta \) at point F by \( \mu \). Denoting the slope of \( u_y \) with respect to \( x \) (Fig. 3) at \( F \) by \( -u_{y|F \beta} \), we have

\[ \omega_1(\mu, \mu) = \frac{\partial u_y}{\partial x} (\alpha = \beta = \mu) = \frac{\sqrt{2}}{2} \left( \frac{\partial u_y}{\partial \xi} + \frac{\partial u_y}{\partial \eta} \right)_{\alpha=\beta=\mu} = \]

\[ \frac{\sqrt{2}}{2} \left( \frac{\partial u_y}{\partial \alpha} \frac{\partial \alpha}{\partial \xi} + \frac{\partial u_y}{\partial \beta} \frac{\partial \beta}{\partial \xi} + \frac{\partial u_y}{\partial \alpha} \frac{\partial \alpha}{\partial \eta} + \frac{\partial u_y}{\partial \beta} \frac{\partial \beta}{\partial \eta} \right)_{\alpha=\beta=\mu}. \]

Due to \( \alpha = \beta \), in (12) we have \( \varphi = 0 \) and \( A = B \) [cf. (7)], and then with the help of (8) the relation (12) reduces to

\[ \omega_1(\mu, \mu) = \frac{1}{2A(\mu, \mu)} \left( \frac{\partial u_y}{\partial \alpha} + \frac{\partial u_y}{\partial \beta} \right)_{\alpha=\beta=\mu}. \]

Both derivatives of \( w_2 \) in (11) with respect to \( \alpha \) and \( \beta \) vanish at \( \alpha = \beta \). Hence (11) and (13) imply

\[ \omega_1(\mu, \mu) = -\frac{k\beta}{A(\mu, \mu)} \left( \frac{\partial u_y}{\partial \alpha} + \frac{\partial u_y}{\partial \beta} \right)_{\alpha=\beta=\mu}, \]

where by (11), using \( I_0' = I_1 \), we have