Optimal design of cylindrical shells

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Abstract In this paper, two types of problems of the optimal design of cylindrical shells with arbitrary axisymmetrical boundary conditions and distributed load, under the condition of the volume being constant, are discussed. These problems involve the minimax deflection and minimal compliancy of a cylindrical shell. Expressions of the objective function can be obtained by a stepped reduction method. In minimizing the maximum deflection, the position of the maximum deflection from the previous iteration is used as the next one. This procedure converges (Avriel 1976). Several examples are provided to illustrate the method.

1 Introduction

Problems of optimal design with respect to a continuous elastic body are very important in both theory and application in the field of modern optimization. As a matter of fact, there are very few papers on the optimal design of shells because the governing equations are very complex (Haug 1980). Here we present an effective way to optimally design a thin cylindrical elastic shell, that can determine the thickness functions which cause the minimax deflection or minimal compliancy of the shell, under the condition of the volume being constant and the middle surface shape being defined. In these problems, the explicit formulations of the objective function cannot be determined by traditional methods, which leads to many computational difficulties. The stepped reduction method (Yu and Yeh 1988; Yeh and Ji 1989) can give the solution of the deflection of cylindrical shells with variable thickness; further the explicit expressions of the objective function can be obtained. The expressions are suitable for the arbitrary axisymmetrical boundary conditions and distributed loads. The problems of optimal design are reduced to a nonlinear programming problem with an equality constraint.

2 Solution of the axisymmetrical deflection of cylindrical shells

Consider the thin cylindrical elastic shell shown in Fig. 1, with the axisymmetrical variable thickness function $H(x)$, length $L$, radius $R$, elastic constants $E$, $\mu$ and arbitrary radial axisymmetrical distributed load $P(x)$. Divide the shell into $n$ elements as shown in Fig. 2. If each shell element is small enough, it can be considered as having uniform thickness and being acted on by a uniformly distributed load. Suppose the $i$-th element has the length $L_i$, thickness $H_i$, distributed radial load $P_i$, local variable $X_i$, $0 \leq X_i \leq L_i$ (lower section of the element $X_i = 0$ and upper section $X_i = L_i$). Then, the differential equation for the radial deflection $W_i$ of the $i$-th shell element is

$$\frac{d^4 W_i}{dx_i^4} + 4 \cdot K_i W_i(x_i) = P_i/D_i,$$

where $D_i = E \cdot H^3_{i}/12(1 - \mu^2)$ is the radial stiffness and $K_i = 3(1 - \mu^2)/(RH_i)^2$. According to Huang and Liang (1983), the solution of (1) can be written as

$$W_i(X_i, H) = W_i(O, H)F_{1i}(X_i, H_i) +$$

$$+W_i^{(1)}(O, H)F_{2i}(X_i, H_i) + M_i(O, H)F_{3i}(X_i, H) +$$
where \( H = (H_1, H_2, \ldots, H_N); W_i(O, H), W^{(1)}_i(O, H), M_i(O, H) \) and \( Q_i(O, H) \) are the deflection, slope, bending moment and shear force, respectively, at \( X_i = 0 \). According to Haug (1980)

\[
F_{1i} = CH(\lambda_i X_i) \cdot \cos(\lambda_i X_i),
\]

\[
F_{2i} = [CH(\lambda_i X_i) \cdot \sin(\lambda_i X_i) + SH(\lambda_i X_i) \cdot \cos(\lambda_i X_i)]/(2\lambda_i),
\]

\[
F_{3i} = SH(\lambda_i X_i) \cdot \sin(\lambda_i X_i)/(2\lambda_i^2),
\]

\[
F_{4i} = [CH(\lambda_i X_i) \cdot \sin(\lambda_i X_i) - SH(\lambda_i X_i) \cdot \cos(\lambda_i X_i)]/(4\lambda_i^3),
\]

\[
F_{hi} = [1 - F_{1i}(\lambda_i X_i)] \cdot P_i/(4\lambda_i^2).
\]

To facilitate deduction and numerical computation, let

\[
x_i = xi/L, \quad h_i = Hi/H_u, \quad w_i = W_i/L,
\]

\[
m_i = M_i/L/D_u, \quad q_i = Q_iL^2/D_u, \quad P_i = P_iL^3/D_u,
\]

\[
D_u = E \cdot H_u^3/[12 \cdot (1 - \mu^2)],
\]

where \( H_u \) denotes the thickness of the uniform shell with given volume \( V_0 \). Then (2) can be written as

\[
w_i(x_i, h) = w_i(O, h)f_{1i}(x_i, h_i) + w^{(1)}_i(O, h)f_{2i}(x_i, h_i) + \ldots
\]

\[
+ m_i(O, h)f_{3i}(x_i, h_i) + q_i(O, h)f_{4i}(x_i, h_i) + f_{hi}(x_i, h_i),
\]

(3)

where \( w_i(O, h), w^{(1)}_i(O, h), m_i(O, h) \) and \( q_i(O, H) \) denote dimensionless deflection, slope, bending moment and shear forces, respectively, at \( x_i = 0 \). The functions \( f_{ki}(x_i, h_i) \) are the corresponding dimensionless functions of \( F_{ki}(X_i, H_i), \) \( k = 1, 2, 3, 4, 5 \) for \( x_i = 0 \). Thus \( w_i(x_i, h) \) is an explicit formulation of \( x_i, h_i \). Let

\[
S_i(x_i, h) = [w_i(x_i, h), w^{(1)}_i(x_i, h), m_i(x_i, h), q_i(x_i, h)]^T.
\]

According to the relations of deflection, slope, bending moment and shear force, we have

\[
S_i(x_i, h) = T_i(x_i, h_i) \cdot S_i(O, h) + U_i(x_i, h_i),
\]

(6)

where \( T_i(x_i, h_i) \) and \( U_i(x_i, h_i) \) denote a \( 4 \times 4 \) matrix and a \( 4 \times 1 \) matrix, respectively, which consist of \( f_{ki}(x_i, h_i), (k = 1, 2, 3, 4, 5) \) and their derivatives. Continuity conditions at the junction of two neighbouring elements must be satisfied, so

\[
S_i(O, h) = S_{i-1}(l_{i-1}, h), \quad (i = 2, 3, \ldots, n),
\]

(7)

where \( l_i = L_i/L, \) \( (i = 1, 2, \ldots, n) \). Thus, we have

\[
S_i(x_i, h) = TT_i(x_i, h_1, h_2, \ldots, h_i)S_1(O, h) + \ldots + UU_i(x_i, h_1, h_2, \ldots, h_i), \quad (i = 1, 2, \ldots, n),
\]

(8)

where

\[
TT_i(x_i, h_i) = T_i(x_i, h_i),
\]

\[
TT_i(x_i, h_1, h_2, \ldots, h_i) = T_i(x_i, h_i)TT_j(l_j, h_j),
\]

\[
(i = 2, 3, \ldots, n), \quad (j = 1, 2, \ldots, i - 1),
\]

\[
UU_i(x_i, h_1) = U_1(x_i, h_1),
\]

\[
UU_i(x_i, h_1, h_2, \ldots, h_i) = U_i(x_i, h_i) + \ldots + T_i(x_i, h_i)[U_{i-1}(l_{i-1}, h_{i-1}) + \sum U_j(l_j, h_j)TT_k(l_k, h_k)],
\]

(9)

\[
(i = 2, 3, \ldots, n), \quad (j = 1, 2, \ldots, i - 2), \quad (k = 1, 2, \ldots, j - 1).
\]

Then \( TT_i(x_i, h_1, \ldots, h_i) \) are the explicit formulation of \( x_i, h_i \). When \( i = n \) and \( x_n = L_n \), equation (8) will give \( w_n(h), w^{(1)}_n(h), m_n(h) \) and \( q_n(h) \). If the boundary conditions of the shell are given, the explicit expressions of the deflection, slope, bending moment and shear force at any point of the shell can be obtained from (8). Because the explicit expression of the deflection of the shell with respect to \( h \) can be obtained from (8), the explicit expression of compliancy with respect to \( h \) can also be obtained. Thus the derivatives of deflection and compliancy with respect to \( h \) can be easily obtained.

3 Optimal design of minimax deflection

The optimal design problem can be stated as follows. Determine the thickness function \( H(X) \) which minimizes

\[
\max W[H(X), X], \quad 0 \leq X \leq L,
\]

subject to

\[
\int H(X) \, dX = V_0/2\pi R, \quad H(X) \geq H_{\text{min}}, \quad 0 \leq X \leq L,
\]

(9)

where \( H_{\text{min}} \) is the given minimal thickness. As previously stated, after dividing the shell into \( n \) shell elements and setting a dimensionless transformation, the optimal problem will be:

- determine \( n + 1 \) variables \( h_1, h_2, \ldots, h_n, x \), to minimize

\[
\max w(h_1, h_2, \ldots, h_n, x),
\]

subject to

\[
\sum h_i l_i = 1, \quad h_i \geq h_{\text{min}}, \quad i = 1, 2, \ldots, n,
\]

(10)

where \( h_{\text{min}} = H_{\text{min}}/H_u \).

If the boundary conditions are known and \( h_i(i = 1, 2, \ldots, n) \) are determined, the solution \( w \) can be obtained. To find the point of the maximal deflection, let the length of each element be small enough so that only one stationary point of deflection exists at each element, if it exists. First, determine