An Unsolvable Problem with Products of Matrices

Melven Krom

Department of Mathematics, University of California, Davis, California 95616

Abstract. The recursive unsolvability of a problem concerning products of upper triangular $3 \times 3$ matrices is shown to be an immediate consequence of the unsolvability of the Post Correspondence problem.

Unsolvable problems that are easy to describe and are closely related to practical problems can be good guides for choosing reasonable projects. One of the best examples in linear algebra is the recursively unsolvable mortality problem for $3 \times 3$ matrices (to decide for any finite set of $3 \times 3$ matrices with integer entries, whether the zero matrix is a finite product of members of the set) [3], [1, p. 584]. Here an unsolvable problem concerned with pairs of finite sets of $3 \times 3$ matrices is explained; its relationship to the mortality problem is discussed below.

For any finite set $\mathcal{K}$ of matrices, let $\mathcal{K}^*$ be the closure of $\mathcal{K}$ under the operation of matrix multiplication, and for any $X \in \mathcal{K}^*$ let $l_{\mathcal{K}}(X)$ be the number of factors in the shortest expression of $X$ as a product of members of $\mathcal{K}$.

**Theorem.** There does not exist a recursive procedure for deciding for any two finite sets $\mathcal{M}$, $\mathcal{N}$ of upper triangular $3 \times 3$ matrices with non-negative integer entries, whether there exists $X \in \mathcal{M}^* \cap \mathcal{N}^*$ such that $l_{\mathcal{M}}(X) = l_{\mathcal{N}}(X)$.

**Proof.** The proof consists of a reduction of the known recursively unsolvable Post Correspondence problem [4], [2, p. 193]. A Post Correspondence system consists of a finite sequence $\langle X(1), Y(1) \rangle, \ldots, \langle X(n), Y(n) \rangle$ of ordered pairs of strings $X(i)$ and $Y(i)$, for $i = 1, \ldots, n$, of symbols from an alphabet set $A$. A solution for the system consists of a finite sequence $i_1, \ldots, i_m$ of positive integers all less than $n+1$ and with repetitions of terms allowed so that, as strings of symbols from $A$, $X(i_1)X(i_2) \cdots X(i_m) = Y(i_1)Y(i_2) \cdots Y(i_m)$. There is no recursive procedure for deciding for any Post Correspondence system with a two element alphabet set $A$, whether the system has a solution.

0025/5661/81/0014-0335$01.00
©1981 Springer-Verlag New York Inc.
Let \( \langle X(1), Y(1) \rangle, \ldots, \langle X(n), Y(n) \rangle \) be a Post Correspondence system with a two element alphabet set. Let the digits 2 and 3 correspond, respectively, to the two alphabet symbols. For any ordered pair \( \langle X(i), Y(i) \rangle \) of the system, let \( X(i) \) and \( Y(i) \) be the non-negative integers which are the strings of the digits 2 and 3 that correspond to the strings \( X(i) \) and \( Y(i) \) of the given alphabet symbols with 0 corresponding to the null string. For each integer \( i \) such that \( 1 \leq i \leq n \) let \( C(i) \) be the integer consisting of the digit 2 followed on the right by \( i \) occurrences of the digit 3. Let \( \mathcal{M} \) be the set consisting of \( n \) matrices \( M_1, M_2, \ldots, M_n \) and let \( \mathcal{N} \) be the set consisting of \( n \) matrices \( N_1, N_2, \ldots, N_n \) where the entry in the \( i \)th row and \( j \)th column of \( M_k \) (of \( N_k \)) will be denoted by \( M(i, j, k) \) (by \( N(i, j, k) \)). The matrices are all to be upper triangular with 1 in the first row first column position. For \( 1 \leq k \leq n \) let \( M(1, 2, k) = X(k) \), \( N(1, 2, k) = Y(k) \), \( M(1, 3, k) = N(1, 3, k) = C(k) \), and \( M(2, 3, k) = N(2, 3, k) = 0 \). Also let \( M(2, 2, k) \) be (let \( N(2, 2, k) \) be) \( 10 \) raised to the power equal to the number of nonzero digit occurrences in \( M(1, 2, k) \) (in \( N(1, 2, k) \)) and let \( M(3, 3, k) = N(3, 3, k) = 10^{k+1} \). For example, a product of two matrices from \( \mathcal{M} \) might be

\[
\begin{bmatrix}
1 & 223 & 23 \\
0 & 1000 & 0 \\
0 & 0 & 100
\end{bmatrix}
\begin{bmatrix}
1 & 32 & 233 \\
0 & 100 & 0 \\
0 & 0 & 1000
\end{bmatrix}
= \begin{bmatrix}
1 & 22332 & 23233 \\
0 & 100000 & 0 \\
0 & 0 & 100000
\end{bmatrix}.
\]

In general the entry in the first row third column position represents the sequence of integers \( i_1, i_2, \ldots, i_m \) that could be a solution for the Post system and the entry in the first row second column position represents \( X(i_1)X(i_2)\ldots X(i_m) \) for products from \( \mathcal{M} \) and \( Y(i_1)Y(i_2)\ldots Y(i_m) \) for products from \( \mathcal{N} \). The given Post Correspondence system has a solution if and only if \( \mathcal{M} \times \mathcal{N} \) is nonempty and if \( X \in \mathcal{M} \times \mathcal{N} \) then \( l_\mathcal{M}(X) = l_\mathcal{N}(X) \).

The proof given above also establishes the statement obtained from the theorem by deleting the requirement \( l_\mathcal{M}(X) = l_\mathcal{N}(X) \). That requirement ensures local decidability (for any \( X \in \mathcal{M} \times \mathcal{N} \) only finitely many products of members of \( \mathcal{N} \) need be checked to see if \( X \) is a solution).

The mortality problem mentioned above is more natural and easier to describe than the problem considered here (even with the simplification of omitting the local decidability requirement). Thus, as might be expected, the proof of unsolvability of the mortality problem is more difficult. The technique, used above, of representing alphabet symbols with digits in such a way that concatenation of strings of symbols corresponds to matrix multiplication was first used in the mortality problem [3]. There, additional intricate features were needed to ensure that the zero matrix could only be produced when words represented in two entry positions of a matrix product are identical. In the above proof, words to be compared for the Post Correspondence problem are represented in the same entry position of two different matrix products. In this variation only upper triangular \( 3 \times 3 \) matrices are needed in which only non-negative integers occur as entries.