Value on a Class of Non-Differentiable Market Games

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Abstract: We prove the existence of a (unique) Aumann-Shapley value on the space on non-atomic games $Q^n$ generated by $n$-handed glove games. (These are the minima of $n$ non-atomic mutually singular probability measures.) It is also shown that this value can be extended to a value on the smallest space containing $Q^n$ and $pNA$.

In their book "Values of Non-Atomic Games", Aumann/Shapley [1974, Chap. 3] have discussed the asymptotic approach to the value concept. In particular, they have proved that a three handed glove game which is a minimum of three non-atomic mutually singular probability measures, does not have an asymptotic value. The questions raised by the authors are whether there is an (axomatic) value on the smallest symmetric subspaces $Q^3$ of BV containing such games; and if the value does exist can one extend it to a value on the smallest linear space containing $Q^3$ and $pNA$. (Note that every game in $pNA$ has an asymptotic value [Aumann/Shapley, Theorem F].) In this paper we give positive answers to these two questions.

Let $(I, \mathcal{C})$ be a given measurable space which is isomorphic to $([0,1], \mathcal{B})$ where $\mathcal{B}$ is the $\sigma$-field of Borel sets on $[0,1]$. Let $n$ be a fixed positive integer. Let $Q^n$ be the linear space generated by all games $\nu$ of the form
\[ \nu = \min (\mu_1, \ldots, \mu_n) \]
where $(\mu_1, \ldots, \mu_n)$ is a vector measure with the properties that $\mu_i \in NA^1$ for each $i$, $1 \leq i \leq n$ and if $i \neq j$ then $\mu_i$ and $\mu_j$ are mutually singular. Any two vectors $(\mu_1, \ldots, \mu_n)$ and $\hat{(\mu_1, \ldots, \mu_n)}$ with these properties are isomorphic, i.e., there is an automorphism $\Theta$ of $(I, \mathcal{C})$ such that
\[ (\Theta^* \mu_1, \ldots, \Theta^* \mu_n) = (\hat{\mu}_1, \ldots, \hat{\mu}_n). \]
Therefore, for a given $(\mu_1, \ldots, \mu_n)$ with the above properties, $Q^n$ is the smallest symmetric subspace of BV containing $\min (\mu_1, \ldots, \mu_n)$. For the rest of the paper $(\mu_1, \ldots, \mu_n)$ will be fixed.

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Theorem 1. For each \( n \) there is a unique value \( \phi \) on the space \( Q^n \). \( \phi \) satisfies

\[
\phi(\min (\mu_1, \ldots, \mu_n)) = \frac{\mu_1 + \cdots + \mu_n}{n}.
\]

Proposition 19.7 of Aumann/Shapley [1974, p. 139] asserts that a value on \( Q^3 \) if it exists must satisfy the same equation as \( \phi \) above. Therefore by additivity and symmetry this value must be the unique value on \( Q^3 \). Using the same arguments as in the proof of Proposition 19.7 it is clear that this result holds in general for any positive integer \( n \), i.e., if a value on \( Q^n \) exists it must be unique. Thus we have to prove only the existence part of Theorem 1 above. The proof of this part is based upon the following lemma.

Lemma 2. Let \( v = \min (\mu_1, \ldots, \mu_n) \). For each \( i, 1 \leq i \leq m \), let \( a_i \in E^1 \) and let \( \Theta_i \)

be an automorphism of \((I, C)\). If \( \sum_{i=1}^{m} a_i \Theta_i^*v \) is a monotonic game then

\[
\sum_{i=1}^{m} a_i \Theta_i^* (\mu_1 + \cdots + \mu_n)/n \geq 0.
\]

Using Lemma 2, the proof of Theorem 1 is as follows:

Any game in \( Q^n \) is of the form \( \sum_{i=1}^{m} a_i \Theta_i^*v \). Define \( \phi : Q^n \to FA \) by

\[
\phi(\sum_{i=1}^{m} a_i \Theta_i^*v) = \sum_{i=1}^{m} a_i \Theta_i^* \left( \frac{\mu_1 + \cdots + \mu_n}{n} \right).
\] (1)

First we have to prove that \( \phi \) is a well defined operator i.e., if \( \sum_{i=1}^{m} a_i \Theta_i^*v = \sum_{j=1}^{n} b_j \Theta_j^*v \) then \( \phi(\sum_{i=1}^{m} a_i \Theta_i^*v) = \phi(\sum_{j=1}^{n} b_j \Theta_j^*v) \). By (1) it is enough to prove that

\( \sum_{i=1}^{m} a_i \Theta_i^*v = 0 \Rightarrow \phi(\sum_{i=1}^{m} a_i \Theta_i^*v) = 0 \). Indeed, if \( \sum_{i=1}^{m} a_i \Theta_i^*v = 0 \) then both, \( \sum_{i=1}^{m} a_i \Theta_i^*v \) and

\(- \sum_{i=1}^{m} a_i \Theta_i^*v \) are monotonic games, therefore by Lemma 2 \( \phi(\sum_{i=1}^{m} a_i \Theta_i^*v) \) and \(- \phi(\sum_{i=1}^{m} a_i \Theta_i^*v) \)

are non negative which implies \( \phi(\sum_{i=1}^{m} a_i \Theta_i^*v) = 0 \). The fact that \( \phi \) is linear, symmetric and efficient follows immediately from (1) and the fact the \( \phi \) is positive is exactly Lemma 2. Thus \( \phi \) is a value on \( Q^n \).

Proof of Lemma 2. Assume that \( \sum_{i=1}^{m} a_i \Theta_i^*v \) is a monotonic game. Let

\( \Pi = (I_1, I_2, \ldots, I_n) \) be a partition of \( I \) (i.e., \( I_k \cap I_j = \emptyset \) if \( k \neq j \), \( \bigcup_{j=1}^{n} I_j = I \) and \( I_j \in C \)), such that \( I_j \) is a support of \( \mu_j \). For each \( i, 1 \leq i \leq m \),

\( \Theta_i^{-1} \Pi = (\Theta_i^{-1} I_1, \ldots, \Theta_i^{-1} I_n) \)

is\(^2\) a partition of \( I \) whose elements are the supports of the vector measure \((\Theta_i^* \mu_1, \ldots, \Theta_i^* \mu_n)\).

\(^2\) For each \( S \in C, \Theta_i^{-1} S = \{ x \in I \mid \Theta_i x \in S \} \).