Value on a Class of Non-Differentiable Market Games

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Abstract: We prove the existence of a (unique) Aumann-Shapley value on the space on non-atomic games $Q^n$ generated by $n$-handed glove games. (These are the minima of $n$ non-atomic mutually singular probability measures.) It is also shown that this value can be extended to a value on the smallest space containing $Q^n$ and $pNA$.

In their book "Values of Non-Atomic Games", Aumann/Shapley [1974, Chap. 3] have discussed the asymptotic approach to the value concept. In particular, they have proved that a three handed glove game which is a minimum of three non-atomic mutually singular probability measures, does not have an asymptotic value. The questions raised by the authors are whether there is an (axomatic) value on the smallest symmetric subspaces $Q^3$ of $BV$ containing such games; and if the value does exist can one extend it to a value on the smallest linear space containing $Q^3$ and $pNA$. (Note that every game in $pNA$ has an asymptotic value [Aumann/Shapley, Theorem F].) In this paper we give positive answers to these two questions.

Let $(I, C)$ be a given measurable space which is isomorphic to $([0,1], B)$ where $B$ is the $\sigma$-field of Borel sets on $[0,1]$. Let $n$ be a fixed positive integer. Let $Q^n$ be the linear space generated by all games $v$ of the form

$$v = \text{min}(\mu_1, \ldots, \mu_n)$$

where $(\mu_1, \ldots, \mu_n)$ is a vector measure with the properties that $\mu_i \in NA$ for each $i$, $1 \leq i \leq n$ and if $i \neq j$ then $\mu_i$ and $\mu_j$ are mutually singular. Any two vectors $(\mu_1, \ldots, \mu_n)$ and $(\hat{\mu}_1, \ldots, \hat{\mu}_n)$ with these properties are isomorphic, i.e., there is an automorphism $\Theta$ of $(I, C)$ such that

$$(\Theta^* \mu_1, \ldots, \Theta^* \mu_n) = (\hat{\mu}_1, \ldots, \hat{\mu}_n).$$

Therefore, for a given $(\mu_1, \ldots, \mu_n)$ with the above properties, $Q^n$ is the smallest symmetric subspace of $BV$ containing $\text{min}(\mu_1, \ldots, \mu_n)$. For the rest of the paper $(\mu_1, \ldots, \mu_n)$ will be fixed.

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Theorem 1. For each $n$ there is a unique value $\varphi$ on the space $Q^n$. $\varphi$ satisfies
\[ \varphi(\min(\mu_1, \ldots, \mu_n)) = \frac{\mu_1 + \cdots + \mu_n}{n}. \]

Proposition 19.7 of Aumann/Shapley [1974, p. 139] asserts that a value on $Q^3$ if it exists must satisfy the same equation as $\varphi$ above. Therefore by additivity and symmetry this value must be the unique value on $Q^3$. Using the same arguments as in the proof of Proposition 19.7 it is clear that this result holds in general for any positive integer $n$, i.e., if a value on $Q^n$ exists it must be unique. Thus we have to prove only the existence part of Theorem 1 above. The proof of this part is based upon the following lemma.

Lemma 2. Let $v = \min(\mu_1, \ldots, \mu_n)$. For each $i$, $1 \leq i \leq m$, let $a_i \in E^1$ and let $\Theta_i$ be an automorphism of $(I, C)$. If $\sum_{i=1}^{m} a_i \Theta_i^* v$ is a monotonic game then
\[ \sum_{i=1}^{m} a_i \Theta_i^* \left( (\mu_1 + \cdots + \mu_n)/n \right) > 0. \]

Using Lemma 2, the proof of Theorem 1 is as follows:

Any game in $Q^n$ is of the form $\sum_{i=1}^{m} a_i \Theta_i^* v$. Define $\varphi : Q^n \to FA$ by
\[ \varphi(\sum_{i=1}^{m} a_i \Theta_i^* v) = \sum_{i=1}^{m} a_i \Theta_i^* \left( \frac{\mu_1 + \cdots + \mu_n}{n} \right). \tag{1} \]

First we have to prove that $\varphi$ is a well defined operator i.e., if $\sum a_i \Theta_i^* v = \sum b_i \Theta_i^* v$ then $\varphi(\sum a_i \Theta_i^* v) = \varphi(\sum b_i \Theta_i^* v)$. By (1) it is enough to prove that $\sum a_i \Theta_i^* v = 0 \Rightarrow \varphi(\sum a_i \Theta_i^* v) = 0$. Indeed, if $\sum a_i \Theta_i^* v = 0$ then both, $\sum a_i \Theta_i^* v$ and $-\sum a_i \Theta_i^* v$ are monotonic games, therefore by Lemma 2 $\varphi(\sum a_i \Theta_i^* v)$ and $-\varphi(\sum a_i \Theta_i^* v)$ are non negative which implies $\varphi(\sum a_i \Theta_i^* v) = 0$. The fact that $\varphi$ is linear, symmetric and efficient follows immediately from (1) and the fact the $\varphi$ is positive is exactly Lemma 2. Thus $\varphi$ is a value on $Q^n$.

Proof of Lemma 2. Assume that $\sum_{i=1}^{m} a_i \Theta_i^* v$ is a monotonic game. Let
\[ \Pi = (I_1, I_2, \ldots, I_n) \]
be a partition of $I$ (i.e., $I_k \cap I_j = \emptyset$ if $k \neq j$, $\bigcup_{j=1}^{n} I_j = I$ and $I_j \in C$), such that $I_j$ is a support of $\mu_j$. For each $i$, $1 \leq i \leq m$,
\[ \Theta_i^{-1} \Pi = (\Theta_i^{-1} I_1, \ldots, \Theta_i^{-1} I_n) \]
is\footnote{For each $S \in C$, $\Theta_i^{-1} S = \{ x \in I \mid \Theta_i x \in S \}$.} a partition of $I$ whose elements are the supports of the vector measure $(\Theta_i^* \mu_1, \ldots, \Theta_i^* \mu_n)$.

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