A Note on Bounding $k$-Terminal Reliability$^1$

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Abstract. A generalization of a theorem of Lomonosov and Polesskii is proved, which provides a novel method for determining upper bounds on the probability that a graph contains a Steiner tree ($k$-terminal reliability).

Key Words. $k$-Terminal reliability, Steiner tree, Reliability bound, Graph transformation, Graph reduction.

1. Preliminaries. Consider a communications network in which hosts communicate with each other via bidirectional point-to-point links. A process at one host may require files or utilities at a number of other hosts, and hence must establish a multipoint communication involving these other hosts, and possibly using others as intermediates whose role is simply to forward messages. In the event that the communications links are prone to failure, we are interested in determining the probability that the required multipoint communication can be established. We model this situation in a simple way initially, by considering just the connection probability, and ignoring for the most part issues of performance.

Let $G = (V, E)$ be an undirected graph. Each edge $e \in E$ has a known probability $p_e$ of operation. Let $K \subseteq V$ be a set of target vertices. These represent the hosts required to participate in the multipoint communication. A Steiner tree for $K$ in $G$ is an acyclic connected subgraph of $G$ in which every vertex of degree 1 (leaf) is in $K$. (No stipulation is made that the Steiner tree is the one with fewest edges.) A Steiner tree is a minimal subgraph whose operation guarantees the feasibility of making the desired multipoint communication.

In a Steiner tree, all leaves are target vertices; interior vertices may be target or nontarget vertices. A nontarget vertex in the Steiner tree is a Steiner vertex.

The $k$-terminal reliability of $G$ with respect to $K$, denoted $\text{Rel}(G; K)$, is the probability that a subgraph of $G$ formed by choosing each edge $e$ with probability $p_e$ independently contains a Steiner tree for $K$. The $k$-terminal reliability provides the probability that the desired multipoint communication can be made.

Two special cases are of particular interest. When $K = \{s, t\}$ we obtain two-terminal reliability, and when $K = V$ we obtain all-terminal reliability. Unfortunately, computing either the two-terminal or the all-terminal reliability of a general graph is $\#P$-complete [10], [11], and hence we do not expect to find efficient

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algorithms for these problems. As a result, much effort has been invested in coping with this complexity.

There is an extensive literature on efficient methods for bounding reliability of networks; see [3] and [5] for an overview. Even a cursory examination of the available techniques shows a fundamental difficulty. While many strategies apply to two-terminal reliability, and many apply to all-terminal reliability, their generalization to $k$-terminal reliability appears ineffective. This results primarily from the fact that many basic problems involving Steiner trees are NP-hard, for example: finding a minimum Steiner tree [7], edge-packing Steiner trees [4], and edge-packing Steiner cuts [4].

We generalize a theorem of Lomonosov and Polesskii in order to devise an efficient method for computing upper bounds on $k$-terminal reliability using upper bounds on two-terminal reliability.

2. Upper Bounds. We prove a simple theorem about a graph transformation, which has some strong consequences. Let $G = (V, E)$ be a graph in which edge $e \in E$ operates with probability $p_e$, and let $K \subseteq V$. For $e \in E$ and $t \in K$, let $G|(t, e)$ be the result of deleting edge $e = \{x, y\}$ and adding two edges, $e_0 = \{x, t\}$ and $e_1 = \{t, y\}$, each having probability $p_e$.

**Theorem 2.1.** For $e = \{x, y\} \in E$ and $t \in K$, $\text{Rel}(G; K) \leq \text{Rel}(G|(t, e); K)$.

**Proof.** $G|(t, e)$ has twice as many subgraphs as $G$. We associate with each operational subgraph $X$ of $G$ two operational subgraphs of $G|(t, e)$, so that the probability of obtaining $X$ in $G$ equals that of obtaining one of these two subgraphs in $G|(t, e)$.

If $e$ is not in $X$, we associate $X$ and $X \cup \{e_1\}$ with $X$. If $e \in X$ and $X \setminus \{e\}$ is operational, we associate $X \setminus \{e\} \cup \{e_0\}$ and $X \setminus \{e\} \cup \{e_0, e_1\}$ with $X$. Lastly, if $e \in X$ and $X \setminus \{e\}$ is failed, consider the subgraph $X \setminus \{e\}$. The target vertices lie in two components $S_x$ and $S_y$, with $x \in S_x$ and $y \in S_y$. If $t \in S_x$, associate with $X$ the two subgraphs $X \setminus \{e\} \cup \{e_1\}$ and $X \setminus \{e\} \cup \{e_0, e_1\}$. If $t \in S_y$, associate with $X$ the two subgraphs $X \setminus \{e\} \cup \{e_0\}$ and $X \setminus \{e\} \cup \{e_0, e_1\}$.

Since $e_0$ and $e_1$ both have probability $p_e$, the probability of obtaining one of the two subgraphs associated with $X$ in $G|(t, e)$ equals that of obtaining $X$ in $G$. Moreover, no subgraph of $G|(t, e)$ is associated with two distinct subgraphs of $G$, and hence the inequality follows. 

To apply this result to the case $K = V$ (all-terminal reliability), we proceed as follows. The cut-tree of a graph $G = (V, E)$ is a tree $T = (V, F)$ in which each edge $f \in F$ has an integer weight; for every pair $v, w \in V$, the size of a minimum edge cutset in $G$ equals the minimum weight on the unique $v, w$-path of $T$ (see [6]). Using this definition and Theorem 2.1, we obtain a simple proof of the Lomonosov–Polesskii theorem [9]: