A Geometric View of Parametric Linear Programming

Ilan Adler\(^2\) and Renato D. C. Monteiro\(^3\)

**Abstract.** We present a new definition of optimality intervals for the parametric right-hand side linear programming (parametric RHS LP) problem \( \varphi(\lambda) = \min \{c^T x | Ax = b + \lambda \vec{b}, x \geq 0\} \). We then show that an optimality interval consists either of a breakpoint or the open interval between two consecutive breakpoints of the continuous piecewise linear convex function \( \varphi(\lambda) \). As a consequence, the optimality intervals form a partition of the closed interval \( \{\lambda; |\varphi(\lambda)| < \infty\} \). Based on these optimality intervals, we also introduce an algorithm for solving the parametric RHS LP problem which requires an LP solver as a subroutine. If a polynomial-time LP solver is used to implement this subroutine, we obtain a substantial improvement on the complexity of those parametric RHS LP instances which exhibit degeneracy. When the number of breakpoints of \( \varphi(\lambda) \) is polynomial in terms of the size of the parametric problem, we show that the latter can be solved in polynomial time.

**Key Words.** Parametric linear programming, Sensitivity analysis, Postoptimality analysis, Linear programming.

1. **Introduction.** The subject of this paper is to study the parametric right-hand side linear programming (parametric RHS LP) problem as follows:

\((P_\lambda) \min \{c^T x | Ax = b + \lambda \vec{b}, x \geq 0\},\)

where \( A \) is an \( m \times n \) matrix and \( b, \vec{b}, \) and \( c \) are vectors of dimensions \( m, m, \) and \( n, \) respectively. The parametric RHS LP problem \( (P_\lambda), \lambda \in \mathbb{R}, \) consists of solving each linear programming (LP) problem \( (P_\lambda) \) for all values of \( \lambda \in \mathbb{R} \) (or for \( \lambda \) in a certain required interval). If \( \varphi(\lambda) \) denotes the optimal value of \( (P_\lambda) \), it is well known that the function \( \lambda \in \mathbb{R} \rightarrow \varphi(\lambda) \) is a convex piecewise linear continuous function. In view of this property, only a finite amount of information is necessary to solve the parametric RHS LP problem. Basically, it consists of finding the “breakpoints” of \( \varphi(\lambda) \) and an optimal solution of \( (P_\lambda) \) for all breakpoints \( \lambda \).

We present a way of approaching this problem which differs from the usual method based on the simplex method. Our main motivation to look back into this problem was the introduction of new methods for solving LP problems like the ellipsoid method introduced by Khachiyan [7] and the new interior point algorithm presented by Karmarkar [6].

---

1 This research was partially funded by the United States Navy-Office of Naval Research under Contract N00014-87-K-0202. Its financial support is gratefully acknowledged.

2 Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720, USA.

3 Systems and Industrial Engineering Department, University of Arizona, Tucson, AZ 85721, USA.

Received April 26, 1989; revised May 20, 1990. Communicated by Nimrod Megiddo.
The existing method to solve this problem is the parametric RHS LP simplex method which was first discussed by Gass and Saaty [5] a few years after the simplex method was developed by Dantzig. Many textbooks describe this variant of the simplex method. See, for instance, Dantzig [1] and Murty [9]. The theory of sensitivity and parametric analysis both in discrete and continuous linear (and nonlinear) optimization has been the subject of intensive research. For example, the book by Gal [4] contains about 700 references related to sensitivity and parametric analysis.

Both the existing theory of sensitivity and parametric analysis depends crucially on the concept of the optimality (or characteristic) interval associated with an optimal basis, that is, the set of values of \( \lambda \) for which this basis is optimal for the LP problem \((P_\lambda)\).

In this paper we introduce a different definition of optimality intervals and derive an algorithm for solving the parametric RHS LP problem which can be implemented with the aid of any LP solver. As a first step we have to get rid of the concept of basis and introduce another invariant associated with the problem in order to define our optimality intervals. This is done by considering those partitions \((B, N)\), which we call optimal partitions, such that \( B \cup N = \{1, \ldots, n\}, B \cap N = \emptyset \) and \((x_j \geq 0, j \in N)\) and \((A_i^T y \leq c_j, j \in B)\) are, respectively, the set of always-active constraints with respect to the primal optimal face and the dual optimal face of some problem \((P_\lambda)\). We then show that an optimality interval is either an open interval between two consecutive breakpoints of \( \varphi(\lambda) \) or consists of a breakpoint itself. This shows that the real line is covered in a unique way using these optimality intervals. This is in contrast with the basis optimality intervals where even the closed interval determined by two consecutive breakpoints of \( \varphi(\lambda) \) can be covered in many ways with possibly an exponential number of these intervals.

The second step is to provide an algorithm for the parametric RHS LP problem, based on any LP solver, that computes a sequence of optimal partitions and their associated optimality intervals so that at the end we have covered the required interval by these optimality intervals. The approach is the same as in the existing pivot method which successively finds adjacent optimality intervals either going to the left or to the right of the real line. However, our approach solves an LP problem to find the adjacent partition and the corresponding optimality interval.

It is well known that the parametric RHS LP problem cannot be solved in polynomial time due the existence of instances of the problem whose corresponding function \( \varphi(\lambda) \) exhibits an exponential number of breakpoints. One of the main consequences of our algorithm is an affirmative answer to the following related computational complexity issue: Can the parametric RHS LP problem be solved in time polynomially bounded by the size of the input and the number of breakpoints of \( \varphi(\lambda) \)? For nondegenerate problems, the answer to this issue is rather trivial and is provided by the parametric RHS simplex method discussed above. For degenerate problems, we show in Section 4 that the parametric RHS LP problem can be solved in \( O(kZ) \) where \( k \) is the number of breakpoints and \( Z \) is the complexity of solving a single LP problem of the same dimension.

Our paper is organized as follows. In Section 2 we introduce some notation and