There Are Planar Graphs Almost as Good as the Complete Graphs and Almost as Cheap as Minimum Spanning Trees

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Abstract. Let S be a set of n points in the plane. For an arbitrary positive rational r, we construct a planar straight-line graph on S that approximates the complete Euclidean graph on S within the factor \((1 + \frac{1}{r})[\frac{2\pi}{3} \cos(n/6)]\), and it has length bounded by \(2r + 1\) times the length of a minimum Euclidean spanning tree on S. Given the Delaunay triangulation of S, the graph can be constructed in linear time.

Key Words. Planar graph, Minimum spanning tree, Complete graph, Delaunay triangulation, Time complexity.

1. Introduction. Consider a set S of n points in the plane. We would like to design a planar network between the points in S that approximates the complete Euclidean graph on S in the following sense: there exists a constant c such that for any pair of points in S there is a path in the network that connects the points and is of length bounded by c times the straight-line distance between them. Planar networks approximating the complete Euclidean graph can be applied in the design of route networks and transmission networks. They also have potential applications in the design of algorithms and heuristics that use shortest or almost shortest distances in the plane.

Chew showed that the Delaunay triangulation of S in the L₁ metric approximates the complete graph on S within a factor bounded from above by \(\sqrt{10}\) [3]. Then, Dobkin et al. [6] showed that the Delaunay triangulation of S in the L₂ metric gives an approximation of the complete graph on S within a factor bounded from above by \([1 + \sqrt{5}]/2\)π. Keil and Gutwin have recently decreased this upper bound to \(2\pi/3 \cos(\pi/6) \approx 2.42\) [8]. Keil also considered another family of networks on S (drawn in the plane with possible crossings) with the number of edges linear in the size of S, and showed that they can very closely approximate the complete Euclidean graph [7]. Interestingly, Das and Joseph have recently showed that other classical triangulations, such as the greedy triangulation of S and the minimum weight triangulation of S also approximate the complete graph on S [4].

In the design of route or transmission networks, both the goodness of approx-
imating the complete Euclidean graph and the cost of the resulting network are important. In the simplest case, the cost of a planar network is proportional to the total length of its edges. Clearly, if the network is connected, in particular, if it approximates the complete Euclidean graph on $S$, it has length not smaller than that of a minimum Euclidean spanning tree of $S$. Therefore, it is of interest to ask whether there exists a planar straight-line graph on $S$ that:

1. approximates the complete graph on $S$; and
2. has length within a constant factor from the length of a minimum Euclidean spanning tree of $S$.

Clearly, any planar straight-line graph that satisfies the above conditions approximates the complete graph and has length within a constant factor from the minimum length of a network that approximates the complete graph. It is not difficult to construct examples of sets $S$ where the Delaunay triangulation of $S$ has length of order $n$ times the length of a minimum Euclidean spanning tree of $S$ (see [9]). Thus, the Delaunay triangulation of $S$ does not satisfy the second requirement above. In this paper, we construct a planar straight-line graph that satisfies both requirements. We do it for an arbitrary, positive approximation parameter $r$ by pruning the Delaunay triangulation of $S$. The graph approximates the complete Euclidean graph within the factor $(1 + 1/r) [2\pi/3 \cos(r\pi/6)]$ and has length not greater than $2r + 1$ times the length of a minimum Euclidean spanning tree of $S$. Given the Delaunay triangulation of $S$, the graph can be constructed in linear time.

Das and Joseph have independently presented similar results in [4]. However, their algorithm for approximating the complete Euclidean graph does not run in linear time and they do not provide any explicit bound on the length of the constructed graph.

2. Preliminaries. We shall use standard set- and graph-theoretic notations and definitions (for instance, see [1]). As for computational geometry notation and definitions, we rely on [10]. Among others, we assume the following conventions:

1. A planar straight-line graph (PSLG for short) is a pair $(V, E)$ such that $V$ is a set of points in the plane and $E$ is a set of nonintersecting, open straight-line segments whose endpoints are in $V$. The points in $V$ are called vertices of $G$, whereas the segments in $E$ are called edges of $G$.
2. For a straight-line segment $s$, $|s|$ denotes the length of $s$. For a PSLG $G$, $|G|$ denotes the total length of edges of $G$.
3. Let $v$, $w$ be two vertices of a connected PSLG $G$. The length of a shortest path in $G$ connecting $v$ with $w$ is denoted by $d_G(v, w)$.
4. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two PSLG such that $V_2$ is a subset of $V_1$. Let $c$ be a positive real number. $G_1$ approximates $G_2$ with factor $c$ if for any edge $(v, w)$ of $G_2$, $\text{dist}_{G_1}(v, w)/\text{dist}_{G_2}(v, w) \leq c$.
5. The Delaunay triangulation of a finite set $S$ of points in the plane is denoted by $DT(S)$. The convex hull of $S$ is denoted by $CH(S)$. 