A New Approach to the Morse-Conley Theory and Some Applications (*)

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Summary. We present a new approach to the Morse theory which is based on a generalization of the Conley index to non locally compact spaces. The variant of the Morse theory which we obtain seems suitable for the applications to nonlinear functionals analysis. Some applications are given here; they mainly concern the study of periodic solutions of second order Hamiltonian systems. Other applications are in some quoted papers.

Introduction.

In the first four sections of this paper we present a new approach to the Conley-index theory to non locally-compact spaces (cf. also [B1] and [B2]). The definitions and the theorems are carried out in a fairly large generality as far as this generality does not complicate too much the theory.

We have worked in a metric space and we have not imposed any compactness assumption to the flow even if in the applications a reacher structure needs to be added.

We have chosen this approach because we think that this level of generality helps to understand the underlying structure and also allows to compare this theory with other versions of the Conley index theory in infinite dimensional spaces (cf. the comparison with Rybakowsky [RY] in section 3).

Moreover it is possible that the study of this structure without any compactness will help the analysis of the critical points at infinity in the sense of Bahri (cf. e.g. [BA] and its references).

In the following sections we are interested to the study of critical points of a $C^1$-functional defined on a Banach space (or more in general on a Finsler manifold).

In this framework we are able to write the analogous of the Morse inequalities for $C^1$-functionals whose critical points may not be isolated.

To do this we introduce the concept of $\varepsilon$-Morse covering (cf. Def. 5.11, Th. 5.12 and Th. 5.14).

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In section 6 we consider the study of $C^2$-functionals on a Hilbert manifold equipped with a Riemannian structure. At this level the theory becomes comparable with the classical Morse theory. Thus if the functional is a Morse functional (i.e., all the critical points of $f$ are nondegenerate) we obtain the same results of the Morse theory. If the functional has only isolated, but possible degenerate critical points then we get the results of the Gromoll Meyer theory ([GM], cf. also [CH1]). However our theory does not suppose «a priori» that the critical points are isolated, since also in this case it is possible to use the concept of $\varepsilon$-Morse covering. This fact will be very useful in the applications of the three following sections.

In Section 7 we present some existence theorems for $C^1$-functionals. Some of these theorems can be obtained via minimax methods (cf. e.g. [R1]). However, the use of the Morse theory gives an extra information which may be essential in some cases. For example we refer to [BF1], [BF3] and [BG] where Theorem 7.5 and Corollary 7.9 are the main tools in solving the problems studied there.

In Section 7 we consider also an elliptic equation for which an existence result is obtained in a relatively easy way (Th. 7.14). We consider this equation as an example for which the Morse theory works better than the minimax theory (of course we do not mean that the Morse theory works always better than the minimax theory; it depends on the problem!).

In Section 8 we apply this theory to equivariant functionals. In this case, the most natural thing to do would be to adapt the equivariant Morse theory (cf. e.g. [BO1] and [PAC] for the nonvariational case) to our theory. This fact would not present particular difficulty.

Instead we exploited the equivariance of the functional in a simpler way using only the theory developed in the previous sections. We did this for two reasons: the first is to keep the paper to a simpler level and to avoid technicalities when it is possible. The second is that this level is sufficient to the applications we considered in the following.

The last four sections are devoted to the applications of the Morse theory to the study of periodic solutions of second order conservative systems.

In sections 7 and 8 we define the Maslov index and the twisting number in such a way that it can be easily related to the Morse index of our theory. The results of this sections are the «easiest translation» in our theory of ideas already existing (cf. e.g. [BO2], [COZ1], [COZ2], [EK], [EKH] and their references). The existence of periodic solutions and their relation to the twisting number is investigated in the last two sections (cf. also [B4]).

1. The generalized Conley index.

Let $M$ be a metric space on which a flow $\gamma$ is defined i.e. a (continuous) map

$$\gamma: \mathbb{R} \times M \to M$$

such that $\gamma(0, x) = x$ and $\gamma(t_1, \gamma(t_2, x)) = \gamma(t_1 + t_2, x)$; $(t_1, t_2 \in \mathbb{R}, x \in M)$. When no ambi-