A Class of Strongly Nonlinear Functional Differential Equations (*)

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consider the following functional differential equation

\begin{equation}
\begin{cases}
    u'(t) + \partial \varphi(u(t)) \ni F(t, u(t), u_t), & 0 < t < T, \\
    u(s) = \nu(s), & \tau \leq s \leq 0,
\end{cases}
\end{equation}

where \( \nu \in C^\infty([\tau, 0]; H) \), \( \nu(0) \in D(\varphi) \) and

\[ \int_\tau^0 \| \partial \varphi(\nu(s)) \|^2 ds < +\infty. \]

The main goal of this paper is to prove an existence result for strong solutions to (1.1). Basic sources of references for this kind of problems are the monograph of Hale [10] and the survey paper of Webb [23].

Although functional differential equations have been intensively studied over the past several years by many authors (see for instance [7-14, 16-20, 23] and the references therein) as far as we know, this is the first attempt to overcome the difficulties encountered when \( F \) is defined merely on \([0, T] \times D(\varphi) \times C^\infty([\tau, 0]; H)\) and \( -F \) lacks monotonicity with respect to its second and third arguments. The possibility of considering such functions \( F \) which, in general, are discontinuous with respect to the induced strong topology on both domain and range, allows us to obtain as particular cases of our main existence theorem new results referring to strongly nonlinear partial differential equations of functional type.

The interesting feature of this class of equations consists in that the state of the system depends not only on its history but also on the history of its \( \epsilon \) diffusion. See the examples in Section 6.

We emphasize that our results seem to be new even in the semilinear case, i.e., when \( \partial \varphi \) is linear.

The method of proof which is partially inspired from [15] is mainly based on a fixed point theorem due to Arino, Gautier and Penot [1] and rests heavily on a deep regularity result due to Brezis [3].

The paper is divided into six Sections, the second one being mainly devoted to some notations, definitions and results widely used in all that follows. In Section 3 we state our main result, while Section 4 is merely concerned with its complete proof. Section 5 contains two results concerning the continuation of the solutions, while in the last Section 6 we analyze some examples in order to emphasize the effectiveness of the abstract results.

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2. - Preliminaries.

Let \( H \) be a real Hilbert space with inner product \((\cdot, \cdot)\) and norm \( \|\cdot\| \) and let \( \varphi: H \to [0, +\infty) \) be a proper, l.s.c., convex function. Set

\[ D(\varphi) := \{ u \in H; \varphi(u) < +\infty \}, \]

\[ \partial \varphi(u) := \{ v \in H; \varphi(v) - \varphi(u) \geq (v, w - u), w \in H \}. \]