A Zero-Sum Stochastic Game Model of Duopoly

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Abstract: We consider a discrete time zero-sum stochastic game model of duopoly and give a partial characterization of each firm's optimal pricing strategy. An extension to a continuous time model is also discussed.

1. Introduction

Most previous game-theoretic models of oligopoly [Friedman; Levitan/Shubik; Mayberry, et. al; Rasor/Ho; Roemer], model oligopoly as a static process. The major exception to this rule is the work of Kirman/Sobel [1974] which models the problem of determining an optimal inventory level as a non-zero sum stochastic game [Rogers; Sobel, 1971].

Our paper is an attempt to develop a zero-sum stochastic game [Bewley/Kohlberg; Shaply] model of the problem of setting a price for a product in a two firm industry. The prices charged by both firms and their current market positions are assumed to influence (in a probabilistic fashion) the future market positions of each firm. This allows us to balance the immediate benefits gained by charging a high price against the future loss of customers caused by charging a high price.

In section 2, our model is introduced and the necessary notation is developed. Sections 3 and 4 give a partial characterization of the optimal policies for each firm. Finally, section 5 introduces a continuous time zero-sum stochastic game model of duopoly and extends the results of sections 3 and 4 to this model.

2. Model Formulation

We consider an industry consisting of two firms (referred to as firms 1 and 2) that produce the same product. The two firms are competing for \(N\) customers and at the beginning of any period \(t (t = 1, 2, \ldots )\) firm 1 is assumed to "control" \(i\) customers while firm 2 "controls" \(N - i\) customers. (If firm \(i\) controls a customer, the customer's purchase, if any, must be made from firm \(i\).) During a period each firm must choose the
price at which they wish to sell their product. Firm 1 can choose from \( l \) prices
\( (p_1 < p_2 < \ldots < p_l) \) while firm 2 can choose from \( m \) prices
\( (p_1^* < p_2^* < \ldots < p_m^*) \).

If firm 1 controls \( i \) customers at the beginning of period \( t \), firm 1 chooses price
\( p_r \), and firm 2 chooses price \( p_s^* \), the following sequence of events ensues.

1. Firm 1 earns a profit
\[ iPrD_1 (p_r) = iR_1 (p_r) , \]
where \( D_1 (\cdot) \) measures the dependence on price of the fraction of firm 1’s customers who purchase the product during a period.

2. Firm 2 earns a profit
\[ (N - i) p_s D_2 (p_s) = (N - i) R_2 (p_s) , \]
where \( D_2 (\cdot) \) measures the dependence on price of the fraction of firm 2’s customers who purchase the product during a period.

3. With probability
\[ q_{ij} (p_r, p_s^*) \]
the number of customers controlled by firm 1 at the beginning of period \( t + 1 \) changes to \( f \).

To simplify the presentation, we assume \( R_1 (\cdot) \), \( R_2 (\cdot) \), and each \( q_{ij} (\cdot, \cdot) \) are differentiable functions defined on \([p_1, p_l] \), \([p_1^*, p_m^*] \) and \([p_1, p_1] \times [p_1^*, p_m^*] \) respectively.

For our purposes the \textit{duopoly game} is a two person zero-sum stochastic game (see Bewley/Kohlberg [1976] and Shapley [1953]) which is characterized by the following:

1. A finite state space \( S \).
2. The finite set of actions \( A_i \) available to player \( v \) (\( v = 1, 2 \)) in state \( i \in S \).
3. The reward or payoff (expected or actual) \( r_i (k_1, k_2) \) which is paid from player 2 to player 1 during a period in which the state is \( i \) and player \( v \) chooses action \( k_v \). Rewards are discounted by a factor \( \beta \).
4. A set of transition probabilities \( \{Q_{ij} (k_1, k_2), i, j \in S, k_v \in A_v \} \), where \( Q_{ij} (k_1, k_2) \)
is the probability that the state during period \( t + 1 \) will be \( f \) given that the state during period \( t \) was \( i \) and player \( v \) chose action \( k_v \).
5. The initial state \( i_0 \).

If we assume that each firm wishes to maximize the difference between their \( \beta \)-discounted profit and their opponent’s \( \beta \)-discounted profit (an assumption which was made in Clemhous, et. al. [1971]), our duopoly model may be formulated as a two person zero-sum stochastic game with state space \( S = \{0, 1, \ldots, N\} \), action spaces \( A_1^1 = \{1, 2, \ldots, l\} \) and \( A_1^2 = \{1, 2, \ldots, m\} \), transition probabilities \( Q_{ij} (k_1, k_2) = q_{ij} (k_1, k_2) \) and rewards \( r_i (k_1, k_2) = iR_1 (p_{k_1}) - (N - i) R_2 (p_{k_2}^*) \). Thus the actions of each firm correspond to the prices chosen during a period and “game” \( i \) is played during any period in which firm 1 controls \( i \) customers. The single firm version of this model has been analyzed by Deshmukh/Winston [1977].

Let \( \Delta (\Pi) \) be the set of \textit{stationary strategies} for firm 1 (2). Then \( \delta \in \Delta \) is a set of \( N + 1 \) probability vectors

\[
\delta_0 = (\delta_0 (1), \ldots, \delta_0 (l)) \\
\cdot \\
\cdot \\
\cdot \\
\delta_N = (\delta_N (1), \ldots, \delta_N (l)),
\]