A Zero-Sum Stochastic Game Model of Duopoly

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Abstract: We consider a discrete time zero-sum stochastic game model of duopoly and give a partial characterization of each firm's optimal pricing strategy. An extension to a continuous time model is also discussed.

1. Introduction

Most previous game-theoretic models of oligopoly [Friedman; Levitan/Shubik; Mayberry, et. al; Rasor/Ho; Roemer], model oligopoly as a static process. The major exception to this rule is the work of Kirman/Sobel [1974] which models the problem of determining an optimal inventory level as a non-zero sum stochastic game [Rogers; Sobel, 1971].

Our paper is an attempt to develop a zero-sum stochastic game [Bewley/Kohlberg; Shaply] model of the problem of setting a price for a product in a two firm industry. The prices charged by both firms and their current market positions are assumed to influence (in a probabilistic fashion) the future market positions of each firm. This allows us to balance the immediate benefits gained by charging a high price against the future loss of customers caused by charging a high price.

In section 2, our model is introduced and the necessary notation is developed. Sections 3 and 4 give a partial characterization of the optimal policies for each firm. Finally, section 5 introduces a continuous time zero-sum stochastic game model of duopoly and extends the results of sections 3 and 4 to this model.

2. Model Formulation

We consider an industry consisting of two firms (referred to as firms 1 and 2) that produce the same product. The two firms are competing for N customers and at the beginning of any period t (t = 1, 2, . . . ) firm 1 is assumed to "control" i customers while firm 2 "controls" N − i customers. (If firm i controls a customer, the customer's purchase, if any, must be made from firm i.) During a period each firm must choose the

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price at which they wish to sell their product. Firm 1 can choose from $l$ prices $p_1 < p_2 < \ldots < p_l$ while firm 2 can choose from $m$ prices $p_1^* < p_2^* < \ldots < p_m^*$.

If firm 1 controls $i$ customers at the beginning of period $t$, firm 1 chooses price $p_r$, and firm 2 chooses price $p_s^*$, the following sequence of events ensues.

1. **Firm 1** earns a profit $iR_1 (p_r)$, where $D_1 (\cdot)$ measures the dependence on price of the fraction of firm 1's customers who purchase the product during a period.

2. **Firm 2** earns a profit $(N - i)R_2 (p_s^*)$, where $D_2 (\cdot)$ measures the dependence on price of the fraction of firm 2's customers who purchase the product during a period.

3. With probability $q_i (p_r, p_s^*)$ the number of customers controlled by firm 1 at the beginning of period $t + 1$ changes to $f$.

To simplify the presentation, we assume $R_1 (\cdot), R_2 (\cdot)$, and each $q_i (\cdot, \cdot)$ are differentiable functions defined on $[p_l, p_1], [p_l, p_m^*]$ and $[p_l, p_l] \rightarrow [p_l, p_m^*]$, respectively.

For our purposes the **duopoly game** is a two person zero-sum stochastic game (see Bewley/Kohlberg [1976] and Shapley [1953]) which is characterized by the following:

1. A finite state space $S$.
2. The finite set of actions $A_i$ available to player $v (v = 1, 2)$ in state $i \in S$.
3. The reward or payoff (expected or actual) $r_i (k_1, k_2)$ which is paid from player 2 to player 1 during a period in which the state is $i$ and player $v$ chooses action $k_v$.

Rewards are discounted by a factor $\beta$.

4. A set of transition probabilities $Q_i (k_1, k_2)$, where $Q_i (k_1, k_2)$ is the probability that the state during period $t + 1$ will be $f$ given that the state during period $t$ was $i$ and player $v$ chose action $k_v$.

5. The initial state $i_0$.

If we assume that each firm wishes to maximize the difference between their $\beta$-discounted profit and their opponent's $\beta$-discounted profit (an assumption which was made in Clemhous, et. al. [1971]), our duopoly model may be formulated as a two person zero-sum stochastic game with state space $S = \{0, 1, \ldots, N\}$, action spaces $A_1 = \{1, 2, \ldots, l\}$ and $A_2 = \{1, 2, \ldots, m\}$, transition probabilities $Q_{ij} (k_1, k_2) = q_{ij} (k_1, k_2)$ and rewards $r_i (k_1, k_2) = iR_1 (p_k) - (N - i)R_2 (p_k^*)$. Thus the actions of each firm correspond to the prices chosen during a period and "game" $i$ is played during any period in which firm 1 controls $i$ customers. The single firm version of this model has been analyzed by Deshmukh/Winston [1977].

Let $\Delta$ (II) be the set of stationary strategies for firm 1 (2). Then $\delta \in \Delta$ is a set of $N + 1$ probability vectors

$$\delta_0 = (\delta_0 (1), \ldots, \delta_0 (l))$$

$$\delta_N = (\delta_N (1), \ldots, \delta_N (l)),$$