Completeness and Non-Speciality of Linear Series
on Space Curves and Deficiency
of the Corresponding Linear Series on Normalization (*)

NADIA CHIARLI (TORINO) (**)  

Summary. - Let $\Gamma$ be a curve of $\mathbb{P}^r$ ($r > 3$) of degree $d$, $C$ its normalization and $\mathcal{A} \supseteq I(\Gamma)$ a saturated, homogeneous ideal of $k[X_0, \ldots, X_r]$. In this paper we show that, if $N > 0$ is an integer such that, for $n > N$, the linear series cut out on $\Gamma$ by the hypersurfaces of degree $n$ is complete and non-special, then the deficiency of the linear series cut out on $C$ by the hypersurfaces of $\mathcal{A}_n$, for $n > N$, is independent of $n$ and can be explicitly calculated; this is the case, for instance, when $N = d - r + 1$, and when $N = \sum n_i - r - 1$ (under suitable conditions) if $\Gamma$ is a component of the complete intersection of $r - 1$ hypersurfaces of degrees $n_i$.

Introduction.

Let $\Gamma \subset \mathbb{P}^r$ ($r > 3$) be a reduced, irreducible, non-degenerate, complete curve of degree $d$ and genera $p_a$ and $p_g$, over an algebraically closed field $k$, and let $\nu: C \rightarrow \Gamma$ be its normalization. Put $S = \bigoplus_{n \geq 0} S_n = k[X_0, \ldots, X_r]$, and let $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ be a saturated, homogeneous ideal of $S$, strictly containing the ideal $I$ of $\Gamma$. Denote by $\Delta(\mathcal{A})$ the effective divisor on $C$ corresponding to $\mathcal{A}$, by $D$ the pull-back, via $\nu$, of a generic hyperplane section of $\Gamma$, and let $V = \text{Proj}(S/\mathcal{A})$.

Now, consider the linear series $\Sigma_{\mathcal{A}}$ cut out on $C$ by the hypersurfaces of $\mathcal{A}_n$ outside of $\Delta(\mathcal{A})$; this corresponds to the image of the canonical map

$$\sigma_n: \mathcal{A}_n \rightarrow H^0(C, O_C(nD - \Delta(\mathcal{A})))$$

and the ideal $\mathcal{A}$ is said to be adjoint to $\Gamma$ when $\sigma_n$ is surjective, i.e. $\Sigma_{\sigma_n}$ is complete, for $n > 0$.

A natural question is: given an adjoint ideal $\mathcal{A}$, what is a lower bound for such $n$'s?

The question was answered first by CASTELNUOVO for the so called Castelnuovo ideal (see [1] and [5] for details): he showed in [2] that, in this case, $\sigma_n$ is surjective

(*) Entrata in Redazione il 27 luglio 1984.

(**) Work done in the ambit of G.N.S.A.G.A. of C.N.R. at the Mathematics and Statistics Department, Queen's University, Kingston (Canada), under financial support from the N.S.E.R.C. of Canada, the Italian M.P.I. and the N.A.T.O. Fellowships Scheme Programme.

The author wishes to thank R. LAZARSELD for advice and the Curves Seminar group at Queen's, in particular A. V. GERAMITA and E. DAVIS, for fruitful and stimulating discussions on this subject.
for \( n > d - 2 \), and examples show that Castelnuovo's estimate is optimal for curves in \( \mathbf{P}^d \).

More recently Arbarello-Ciliberto in [1] gave a lower bound for \( n \), when \( \mathcal{A} \) is the Petri adjoint ideal, and Ciliberto-Orecchia for a general adjoint ideal (see [5]).

Moreover, when \( \Gamma \) is non singular and \( \mathcal{A} \) is the whole ring \( \mathcal{S} \), Gruener-Lazarsfeld-Peskine generalized Castelnuovo's result to \( \mathbf{P}^r \), showing that \( \sigma_n \) is surjective for \( n > d - r + 1 \) (see [8]).

In [3] we proved that, for any given ideal \( \mathcal{A} \), there exists a non-negative integer \( B(\mathcal{A}) \) such that, for \( n > B(\mathcal{A}) \), the deficiency \( \lambda_{\mathcal{A}}^{(n)} = \dim \text{coker} \Sigma_{\mathcal{A},n}^{(n)} \) is independent of \( n \), and we have

\[
\lambda_{\mathcal{A}}^{(n)} = \lambda_{\mathcal{A}} = p_a - p_g - (\hat{\beta}_{\mathcal{A}} - \sigma_{\mathcal{A}})
\]

where \( \hat{\beta}_{\mathcal{A}} = \deg \mathcal{A}(\mathcal{A}) \) and \( \sigma_{\mathcal{A}} \) is the number of independent linear conditions imposed by \( \mathcal{V} \) on \( \mathcal{S}_n (n > 0) \).

This gives in particular that, when \( \mathcal{A} \) is adjoint, for \( n > B(\mathcal{A}) \) we have \( \lambda_{\mathcal{A}}^{(n)} = \lambda_{\mathcal{A}} = 0 \) and, when \( \mathcal{A} = \mathcal{S} \), \( \lambda_{\mathcal{S}}^{(n)} = \lambda_{\mathcal{S}} = p_a - p_g \) (see [3], 2.6).

We say in this case that, for \( n > B(\mathcal{A}) \), \( \lambda_{\mathcal{A}}^{(n)} \) stabilizes at \( \lambda_{\mathcal{A}} \).

Now, our bound \( B(\mathcal{A}) \), which improves the bound given by Ciliberto-Orecchia, is equally difficult to compute in concrete cases, and the same is true for the bound given by Arbarello-Ciliberto.

The first aim of this paper was then to find easily computable bounds \( N \) such that \( \lambda_{\mathcal{A}}^{(n)} \) stabilizes at \( \lambda_{\mathcal{A}} \), for \( n > N \); hence, in particular, \( \sigma_n \) is surjective when \( \mathcal{A} \) is adjoint.

More generally, the aim of this paper is to investigate if there is any relationship between a bound which gives the completeness of the linear series \( \Sigma^{(n)} \) cut out on \( \Gamma \) by the hypersurfaces of degree \( n \), and a bound which gives the stabilization of \( \lambda_{\mathcal{A}}^{(n)} \) at \( \lambda_{\mathcal{A}} \).

Section 1 answers the last question raised above: we show that, if \( N \) is a non-negative integer such that, for \( n > N \), \( \Sigma^{(n)} \) is complete and non-special, then, for \( n > N \), \( \lambda_{\mathcal{A}}^{(n)} \) stabilizes at \( \lambda_{\mathcal{A}} \) if \( \mathcal{A} \) satisfies certain cohomological conditions; in particular \( \sigma_n \) is surjective, when \( \mathcal{A} \) is adjoint (see 1.1, 1.2, 1.3).

This is the case e.g. for \( N = d - r + 1 \), as follows easily by [8] (see section 2).

This is also the case for \( N = \sum n_i - r \), or \( N = \sum n_i - r - 1 \) (under certain conditions), when \( \Gamma \) is a component of the complete intersection of \( r - 1 \) hypersurfaces of \( \mathbf{P}^r \) of degrees \( n_1, \ldots, n_{r-1} \) (see 3.4); this bound is clearly much better than the previous one, whenever it is possible to find hypersurfaces through \( \Gamma \) of low degree.

This case is studied in section 3, where we prove a generalization of a theorem of M. Noether ([10]), Gaeta ([9]) and Severi ([7]) which might be of some interest in itself.

In section 4 we study the case when \( \Gamma \) is a curve on a quadric surface of \( \mathbf{P}^3 \).