Some Sensitivity Aspects of Optimal Control Calculations for Economic Systems¹)

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Abstract: In this paper we consider the problem of minimizing a quadratic welfare cost function constrained by a linear macro-econometric model. First we formulate the optimal control problem in a suitable form and specify the solution. Then we consider the sensitivity of the optimal control to changes in the targets and the weights of the welfare cost function. It is shown that the targets should not be selected without reference to the selection of the weights. Next, some simulation experiments, comparing open-loop versus closed-loop controls, are briefly reported on. The effects of changes in the lag structure are also mentioned.

1. Introduction

Traditionally, the optimal control problem has been formulated in continuous time as a matter of choosing a function of time to optimize a criterion function. For a discrete-time model, however, we can regard the optimal control problem as a matter of choosing the values of variables to optimize our criterion function. In this paper we take the latter approach, and apply it to discrete-time macro-econometric models for optimal control calculations. More specifically, the discrete-time optimal control problem is treated numerically as an ordinary mathematical programming problem with equality constraints, the variables being the control values at the various time points and the constraints consisting of the model equations at the corresponding time points. We use an essentially quadratic criterion (welfare cost) function but in the simulations allow for so-called “dead zones”.

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2. Finding the Optimal Control Solution

Let us consider the reduced form of a linear econometric model:

\[ x_t = A_t x_{t-1} + B_t u_t + b_t + e_t, \quad t = 1, \ldots, N. \]  

(1)

The state (column) \( n \)-vector \( x_t \) refers to endogenous variables, and the control \( m \)-vector \( u_t \) to policy variables at the disposition of a decision maker. The coefficient matrices \( A_t \) (\( n \) by \( n \)) and \( B_t \) (\( n \) by \( m \)) (assumed to be known) are usually considered as constant, but as this is not necessary for the further discussion, the time subscripts are retained to make the treatment more general. \( b_t \) accounts for exogenous variables not subject to control, as well as for any constants in the system so that the error term \( e_t \) can be assumed to have zero mean. Although our model is written as a first order system, this formulation covers as well the general \( n \)-th order case (see any text dealing with elementary theory for linear systems, e.g. Jacobs [1974, Ch. 7] or Chow [1975, Ch. 2]). In what follows, we shall put the error term identically equal to zero, \( e_t = 0 \), and consider only the deterministic case. Our results will however carry over to the general stochastic case along the lines of the exposition in Chow [1975, Ch. 7].

To further simplify our exposition, we consider the trajectory \( (\bar{x}_t)_t \), obtained by keeping the controls in (1) at zero for all time periods:

\[ \bar{x}_t = A_t \bar{x}_{t-1} + b_t, \quad t = 1, \ldots, N. \]

(2)

The actual \( x_t \) can then be seen as the sum of this "no-control" \( \bar{x}_t \) and a deviation \( x^*_t \) from \( \bar{x}_t \): \( x_t = \bar{x}_t + x^*_t \), i.e. \( x^*_t = x_t - \bar{x}_t \). Subtracting (2) from (1) we find:

\[ x_t - \bar{x}_t = A_t (x_{t-1} - \bar{x}_{t-1}) + B_t u_t + (b_t - b_t) \]

i.e.

\[ x^*_t = A_t x^*_{t-1} + B_t u_t. \]

(3)

The welfare cost \( J \), to be minimized, is a quadratic function:

\[ J = \sum_{t=1}^{N} (x_t - a_t)' P_t (x_t - a_t) + u'_t Q_t u_t \]

(4)

where \( P_t \) and \( Q_t \) are known, symmetric matrices, \( P_t \) (\( n \) by \( n \)) positive semidefinite and \( Q_t \) (\( m \) by \( m \)) positive definite.\(^3\) This implies that there is a non-negative welfare cost associated with each deviation \( x_t - a_t \) from the given ideal (nominal) state \( a_t \), and a positive welfare cost for each component of \( u_t \) for all periods. Typically, \( P_t \) and \( Q_t \) are going to be diagonal (so that no cross products of states or instruments will enter) and constant (may be with a discounting factor). In fact, even along the diagonal of

\(^3\) A symmetric square matrix \( A \) is positive semidefinite if \( x' A x \geq 0 \) for all \( x \) and positive definite if \( x' A x > 0 \) for all \( x \neq 0 \).