Some Remarks on the Schemes $W_d^r$ (*).

MARC COPPENS

Summary. – Let $X$ be an irreducible smooth projective curve of genus $g$. Let $\phi_d^r(g)$ be the Brill-Noether Number. In this paper we prove some results concerning the schemes $W_d^r$ of special divisors. 1) Suppose $\dim(W_{d-1}^r) = \phi_{d-1}^r(g) > 0$ and $\phi_d^r(g) < g$. If $W_{d-1}^r$ is a reduced (resp. irreducible) scheme, then $W_d^r$ is a reduced (resp. irreducible) scheme. 2) Under certain conditions, if $Z$ is a generically reduced irreducible component of $W_{d-1}^r$ then $Z \oplus W_1^1$ is a generically reduced irreducible component of $W_d^r$. For $r = 1$, we obtain some further results in this direction. 3) As an application of it we are able to prove some dimension theorems for the schemes $W_d^r$.

1. – Introduction.

Let $X$ be a smooth irreducible projective curve of genus $g > 1$ and let $J(X)$ be the jacobian of $X$. This is an abelian variety of dimension $g$ which can be identified with $\text{Pic}^d(X)$, the Picard scheme of the invertible sheaves of degree 0 on $X$. We always make this identification. Let $P_0$ be a fixed base point on $X$ and let $X^{(d)}$ be the $d$-th symmetric product. We have a natural morphism

$$I(d): X^{(d)} \to J(X): D \mapsto [O_X(D - dP_0)]$$

(if $L$ is an invertible sheaf of degree 0 on $X$, then $[L]$ is the corresponding point on $J(X)$). Consider

$$W_d^r = \{x \in J(X): \dim(I(d)^{-1}(x)) > r\} = \{[L] \in \text{Pic}^d(X): h^0(L(dP_0)) > r + 1\}.$$

Those subsets are different from $J(X)$ if $r > d - g$. Those subsets play a central role in the study of special linear systems. The Riemann-Roch Theorem tells us that it is enough to study the case $d < g - 1$. On $W_d^r$ there exists a natural scheme structure. For more details concerning the general theory of the schemes $W_d^r$ we refer to [2], especially Chapter IV.

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Indirizzo dell'A.: Katholieke Industriële Hogeschool der Kempen, Campus H. I. Kempen Kleinhoofdstraat 4, B 2440 Geel, Belgium.
Those schemes $W_d$ are very well-known if $X$ is a general curve. This is the so-called Brill-Noether Theory. We refer to [2], Chapter V, for a summary of the most important results of that theory. If $X$ is an arbitrary curve, the behaviour of the schemes $W_d$ is far from being well-understood. It is the aim of this paper to present some results in this direction.

An important known result is the following.

If $\dim (W_d) > r + 1$ then $W_{d-1} \neq \emptyset$ and $\dim (W_{d-1}) > \dim (W_d) - (r + 1)$.

This is proved in [7] as a consequence of the theory developed in [6]. A consequence of this statement is the following statement.

Suppose $W_d \neq J(X)$ and $d_{d-1}(g) > 0$ (Brill-Noether Number).

If $\dim (W_d) > d_d(g)$ then $\dim (W_{d-1}) > d_{d-1}(g)$.

Hence failure with respect to Brill-Noether behaviour for large $d$ implies failure for $d_d(g, r)$, where

$$d_d(g, r) = \min \{d: d_d(g) > 0\}.$$

By making a closer analysis of the proof of the mentioned result we can push on this philosophy a little bit as follows.

**Result 1** (Theorem 4). — Suppose $\dim (W_{d-1}) = d_{d-1}(g) > 0$ and $d_d(g) < g$.

a) If $W_{d-1}$ is a reduced scheme then $W_d$ is a reduced scheme.

b) If $W_{d-1}$ is an irreducible scheme then $W_d$ is an irreducible scheme.

In the proof of this result we use the description of the tangent space to $W_d$ at a point $x \in W_d \setminus W_{d+1}$ by means of the Petri map (see [2], Chapter IV). Using this description we are able to prove the following fact relating $W_d$ to $W_{d+1}$.

**Result 2** (Theorem 5). — Suppose $Z$ is a generically reduced irreducible component of $W_d \neq J(X)$. Suppose that, for a general point $z$ on $Z$ one has

$$h^1(L_d^2(2dP_0)) \neq 0$$

(i.e., $2D$, where $D$ is a divisor of degree $d$ associated to $z$, is a special divisor). Then $Z \oplus W_d$ is a generically reduced irreducible component of $W_{d+1}$.

(If $A, B \subseteq J(X)$ then $A \oplus B = \{a + b: a \in A$ and $b \in B\}$. We also use $A \ominus B = \{a - b: a \in A$ and $b \in B\}$.)

We prove that the condition on the general point $z$ is always satisfied if $Z \notin W_{d-1} \oplus W_d$ and $\dim (Z) > 2d - g - 2r$ (this follows from Remark 6). In particular the condition in Result 2 is always satisfied if $r = 1$ and $\dim (Z) > d_d(g)$. 