Minima of Some non Convex non Coercive Problems (*).

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Abstract. - We give here an existence result of minimizers for a class of one dimensional integrals of the Calculus of Variations with non convex, non coercive integrands.

1. Introduction and main result.

Let us consider the functional

\begin{equation}
F(u) = \int_0^1 f(x, u(x), u'(x)) \, dx
\end{equation}

defined in the class $\mathcal{W}_p = \{u \in W^{1,p}(0, 1): u(0) = 0, u(1) = \lambda, u' \geq 0 \text{ a.e.}\}$ with $\lambda \in \mathbb{R}_+$ and $p \geq 1$. The integrand $f = f(x, s, \xi)$ is not assumed to be neither coercive nor convex with respect to $\xi$. The closure of $\mathcal{W}_p$ in the (either strong or weak) topology of $W^{1,p}_{loc}(0, 1)$ is given by

\begin{equation}
\overline{\mathcal{W}}_p = \{u \in W^{1,p}_{loc}(0, 1): u(0) \geq 0, u(1) \leq \lambda, u' \geq 0 \text{ a.e.}\},
\end{equation}

where the values $u(0)$ and $u(1)$ are defined by

\begin{align*}
u(0) &= \inf_{x \in (0, 1)} u(x), \\
u(1) &= \sup_{x \in (0, 1)} u(x).
\end{align*}

The extension of $F$ «by lower semicontinuity» from $\mathcal{W}_p$ to $\overline{\mathcal{W}}_p$ is the functional $\overline{F}$ defined for $u \in \overline{\mathcal{W}}_p$ by

\[\overline{F}(u) = \inf_{\{u_k\}} \left\{ \liminf_{k \to \infty} F(u_k); \{u_k\} \subset \mathcal{W}_p, u_k \rightharpoonup_{w}^p u \right\}.\]

Let us precise the hypotheses on the integrand function $f$:

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A1) $f$ is a Carathéodory function on $[0, 1] \times \mathbb{R} \times \mathbb{R}$;

A2) there exist $K > 0$, a convex function $h = h(\xi)$ and continuous functions $a = a(x, s)$ and $b = b(x, s)$ such that for every $x \in [0, 1], s \in \mathbb{R}, \xi \in \mathbb{R}$

i) $a(x, s) h(\xi) - K \leq f(x, s, \xi) \leq a(x, s) h(\xi) + b(x, s),$

ii) $|\xi| \leq h(\xi) \leq L(1 + |\xi|^p), L \in \mathbb{R}^+,$

iii) $a(x, s) \geq 0.$

Then the function $f^{**}$ (which is the greatest function convex with respect to $\xi$ and less than or equal to $f$) satisfies the same assumptions and the lower semicontinuous extension of

$$G(u) = \int_0^1 f^{**}(x, u(x), u'(x)) \, dx, \quad u \in \overline{\mathcal{W}_p}$$

to $\overline{\mathcal{W}_p}$, can be represented as

$$(1.3) \quad \overline{G}(u) = \int_0^1 f^{**}(x, u(x), u'(x)) \, dx \, + \, \int_0^1 \left[ a(0, s) \, ds \, + \, \int_{u(1)}^\lambda \, a(1, s) \, ds \right],$$

where, for simplicity, we set $\bar{h} = h_+ = h_-$,

$$h_\pm = \lim_{\xi \to \pm} \frac{h(\xi)}{\xi},$$

(see [B.-M.], Theorem 2.4).

We are interested in the existence of solutions, for the following problem:

$$(1.4) \quad \min \left\{ \int_0^1 f(x, u(x), u'(x)) \, dx \, + \, \int_0^1 \left[ a(0, s) \, ds \, + \, \int_{u(1)}^\lambda \, a(1, s) \, ds \right], \quad u \in \overline{\mathcal{W}_p} \right\},$$

where $\overline{\mathcal{W}_p}$ is defined by (1.2).

Usually existence for a non convex problem is achieved in two steps: find a minimizer $u_0 \in \overline{\mathcal{W}_p}$ of the relaxed functional (here (1.3)) and then prove that, for such $u_0, f(x, u_0(x), u'_0(x)) = f^{**}(x, u_0(x), u'_0(x))$ a.e. in $\Omega$.

Therefore we need assumptions on $f^{**}$ in order to prove existence of minima of the functional (1.3) in the class (1.2):

B1) $f^{**}$ admits continuous partial derivatives

$$f_{ss}^{**}, f_{s\xi}^{**}, f_{\xi\xi}^{**}, f_{\xi s}^{**}, f_{sss}^{**}, f_{ss\xi}^{**}, f_{s\xi\xi}^{**}, f_{\xi\xi\xi}^{**};$$

B2) there exist an exponent $p \geq 1$ and a function $M: \mathbb{R}^+_+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for $\delta, r > 0$,

$$|f_{s}^{**}(x, s, \xi)| \leq M(\delta, r)(1 + |\xi|^p), \quad \forall (x, s, \xi) \in [\delta, 1 - \delta] \times [-r, r] \times \mathbb{R}.$$