On Linear Isometrics of Cartan Factors in Infinite Dimensions (*).

J. Hervés

Summary. – We obtain the expression of all surjective linear isometries of Cartan factors of types II and III.

The open unit ball $B(\mathcal{H})$ of any $J^{*}$-algebra $\mathcal{H}$ is a bounded circular homogeneous domain. In order to know the group of its analytic automorphisms, $\text{Aut} B(\mathcal{H})$, it therefore suffices to know the group of surjective linear isometries of $\mathcal{H}$. In infinite dimensions, Frenzoni [4] has solved the problem when $\mathcal{H}$ is a Cartan factor of type I. In the present article the solution when $\mathcal{H}$ is a Cartan factor of type II or III is presented. The results obtained can in fact be extended to a wider class of cases including that in which $\mathcal{H} = \ell_2(H, K)$ is the space of compact operators from $H$ to $K$, which has been solved by Arazzi [1] when $H$ and $K$ are separable Hilbert spaces.

1. – Preliminaries.

In this section we recall the definitions and certain results concerning $J^{*}$-algebras given by Harris [7].

Let $H$ and $K$ be complex Hilbert spaces and $\ell(H, K)$ the complex Banach space of bounded linear operators from $H$ to $K$. A $J^{*}$-algebra is a closed complex subspace $\mathcal{H}$ of $\ell(H, K)$ such that for all $A \in \mathcal{H}$, $AA^{*}A \in \mathcal{H}$. If $x \rightarrow \bar{x}$ is a given conjugation in $H$ and $A^{*}x = A^{*}\bar{x}$ for all $x \in H$, then $\mathcal{H}_{I} = \ell(H, K)$, $\mathcal{H}_{II} = \{ A \in \ell(H) : A^{*} = A \}$ and $\mathcal{H}_{III} = \{ A \in \ell(H) : A^{*} = -A \}$ are $J^{*}$-algebras known as Cartan factors of types I, II and III respectively. Any closed complex subspace $\mathcal{H}$ of $\ell(H)$ such that $A^{*} \in \mathcal{H}$ and $A^{*} \in CI$ for all $A \in \mathcal{H}$ is a $J^{*}$-algebra known as a Cartan factor of type IV unless $\dim \mathcal{H} = 2$. Any Cartan factor is closed in the weak operator topology [5].

(*) Entraîné par redaction 13 octobre 1984.

Indirizzo dell'A.: Departamento de Teoría de Funciones, Facultad de Matemáticas, Santiago de Compostela, España.
1.1. Proposition. – Let $\mathfrak{A}$ be a $J^*$-algebra, $A$, $B$, and $C \in \mathfrak{A}$ and $p$ any polynomial. Then

(a) $AB^*C + CB^*A \in \mathfrak{A}$;
(b) $A(B^*A)^* = (AB^*)^*A \in \mathfrak{A}$;
(c) $p(AB^*)C + Cp(B^*A) \in \mathfrak{A}$;
(d) $p(AB^*)Cp(B^*A) \in \mathfrak{A}$.

1.2. Definition. – A $J^*$-isomorphism between $J^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a bounded linear bijection $L: \mathfrak{A} \to \mathfrak{B}$ such that $L(AA^*A) = L(A)L(A)^*L(A)$ for all $A \in \mathfrak{A}$.

1.3. Proposition. – Any $J^*$-isomorphism $L: \mathfrak{A} \to \mathfrak{B}$ is an isometry and commutes with each of the formulas (a)-(d) (1.1). Conversely, any surjective linear isometry is a $J^*$-isomorphism.


In this section we list some properties of operators which can be expressed purely in terms of $J^*$-structures and are thus preserved by $J^*$-isomorphisms [5].

2.1. Definition. – (a) An operator $V$ in a $J^*$-algebra $\mathfrak{A}$ is a partial isometry if and only if $VV^*V =: V$.

Given $A \in \mathfrak{A}$, a partial isometry $V \in \mathfrak{A}$ is said

(b) to cover $A$ if $VV^*A = AV^*V = A$;
(c) to commute with $A$ if $VV^*A = AV^*V$;
(d) to be a unitary element of $\mathfrak{A}$ if covers all $A \in \mathfrak{A}$;
(e) to be a central element of $\mathfrak{A}$ if $V$ commutes with each $A \in \mathfrak{A}$;
(f) to be a minimal element of $\mathfrak{A}$ if for each $A \in \mathfrak{A}$ there exists $\lambda \in C$ such that $V\lambda^*V = \lambda V$;
(g) to be an extreme element of $\mathfrak{A}$ if

$$(I - VV^*)A(I - V^*V) = 0 \quad \text{for all } A \in \mathfrak{A}.\,$$

$A, B \in \mathfrak{A}$ are said to be

(h) orthogonal if $AB^* = 0$ and $B^*A = 0$;
(i) very orthogonal if $AC^*B = 0$ and $BC^*A = 0$ for all $C \in \mathfrak{A}$;
(j) $*$-commutative if $AB^* = BA^*$ and $B^*A = A^*B$;
(k) $C \in \mathfrak{A}$ is indecomposable if are no non-zero orthogonal elements $A, B \in \mathfrak{A}$ such that $A + B = C$. 