Regression by Spline Functions

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Summary: The article deals with the use of Spline Functions, piecewise polynomial functions, in regression models. According to their use in physics and mathematics where one is interested in fitting smooth curves through given fixed points. Poirier [1973] suggests a method estimating those points. Here it is shown that it is possible to estimate the parameters of a Spline directly from the data by the Least Square Estimator. In part 2, the Spline estimation theory given here is applied to a model originally proposed by Barzel [1972]. Here a cubic, quadratic and linear Spline is used as regression functions.

1. Spline Function Estimating Theory

First the abstract definition of a Spline Function is given. $C^n_I$ be the space of n-times continuous differentiable functions at some interval $I$. $I$ may be an infinite interval i.e. $x_0 = -\infty$ and $x_k = \infty$ are allowed. A mesh is a point set $\Delta$ defined as follows:

$$\Delta = \{x_j | x_j \in \mathbb{R}; j = 0, 1, \ldots, k; x_i < x_j, \text{if } i < j\}. \quad (1)$$

Now it is easy to give the definition of a Spline Function. Definition: A Spline Function of degree $n$ on a mesh $\Delta$ denoted by $S^n_\Delta (\xi)$, is a function with the following properties:

$$S^n_\Delta (\xi) \in C^{n-1}_I; \quad I = (x_0, x_k); \quad n = 1, 2, \ldots \quad (2)$$

$$S^n_\Delta (\xi) = P^n_i (\xi) \quad \text{if } \xi \in (x_{i-1}, x_i); \quad i = 1, 2, \ldots, k. \quad (3)$$

Where $P^n_i$ is usually a polynomial of degree $n$ at most. Obviously a Spline is characterized by the fact that there may be discontinuities at the $x_j$ in the $n$-th derivative because a polynomial is clearly an element of $C^n_I$.

Spline Functions are used in mathematics as a curve fitting tool; cf. Ahlberg, Nilson and Walsh [1967]. Poirier [1973] suggested their use in piecewise regression. He showed a method estimating the parameters of Cubic Splines using LS-estimators of the ordinates of the joint points. Unfortunately one gets a linear equation system of not full rank, so that one will have to find plausible linear independent conditions to get a
unique solution. This method is inspired by the common use of Splines in physics, where the problem is to fit a smooth curve through given fixed points. One gets the equation system by evaluating the continuity conditions, but there are none at the borders $x_0$ and $x_k$.

In the following it is shown how to estimate the parameters of the Spline directly from the data.

It is easy to verify that by

$$S^n_\Delta (\xi) = \sum_{j=0}^{n} a_j \xi^j + \sum_{t=1}^{k-1} d_t (\xi - x_t)^n_+$$  \hspace{1cm} (4)

where

$$(\xi - x_t)_+ = \begin{cases} 
0 & \text{if } \xi \leq x_t \\
\xi - x_t & \text{if } \xi > x_t 
\end{cases}$$  \hspace{1cm} (5)

a Spline Function of degree $n$ fulfilling the above definition is given indeed, cf. Powell [1969] p. 398. The mesh $\Delta$ is given, i.e. the $x_t$ are known.

The $a_j$'s and the $d_t$'s are the parameters of the Spline. Apparently the Spline Function is linear in these parameters.

Define now some vectors and matrices:

\[
a = \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_n
\end{bmatrix} \quad \text{a } (n+1 \times 1) \text{ vector}
\]

\[
d = \begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_{k-1}
\end{bmatrix} \quad \text{a } (k-1 \times 1) \text{ vector}
\]

\[
\Xi = \begin{bmatrix}
1 & \xi_1^2 & \xi_1^3 & \ldots & \xi_1^n \\
1 & \xi_2^2 & \xi_2^3 & \ldots & \xi_2^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \xi_m^2 & \xi_m^3 & \ldots & \xi_m^n
\end{bmatrix} \quad \text{a } (m \times n+1) \text{ matrix}
\]