An Extension of the Aumann-Shapley Value Concept to Functions on Arbitrary Banach Spaces

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Abstract: Let X denote a linear space of real valued functions defined on a subset of a Banach space such that X contains E', the dual space of E as a subspace. Given a distinguished vector x₀ in E an x₀-value (on X) is defined to be a projection P from X onto E' which satisfies the following two hypotheses: (VA) (PF)(x₀) = F(x₀) for all F in X; (VB) If T is a continuous isomorphism from E into E such that Tx₀ = x₀ then P(F o T) = (PF) o T for all F in X. The existence and uniqueness of a value is established for two choices of X, one of which is the space of polynomials in functionals on E. The existence and partial uniqueness of a value is established on a third choice for X.

The value of a game having a continuum of players has been defined by Aumann/Shapley [1974] as a positive linear operator which acts between certain spaces of set functions and satisfies certain symmetry conditions. Specifically, let (I, C) denote a measurable space which is isomorphic to the space ([0,1], B) where B denotes the collection of Borel subsets of [0,1]. The space BV consists of all real valued functions u on C of the form u = u⁺ - u⁻ where u⁺ and u⁻ are increasing (i.e. A ⊆ B implies u⁺(A) ≤ u⁺(B)). Let I denote the group of automorphisms of (I, C). A subspace X of BV is called symmetric if u o π is in X for each x in X and each π in I. A value is a linear mapping T from a symmetric subspace X of BV onto the space FA of finitely additive set functions which satisfies three conditions

(AS-1) T is positive: i.e. Tu is increasing whenever u is increasing
(AS-2) T is symmetric: i.e. T(u o π) = (Tu) o π for each π in I and u in X.
(AS-3) T is efficient: i.e. (Tu)(I) = u(I) for each u in X.

It is often required in addition that a value be idempotent, that is, that Tu = u for u in FA. This means that a value T will be a positive linear projection from X onto FA.

The purpose of this paper is to define a generalization of the value in the setting of an arbitrary Banach space (Definition 1.1) and to see to what extent we can derive existence and uniqueness theorems of the type obtained by Aumann/Shapley [1974].
We have three such theorems all stated in section 1 and proved in sections 2 through 4. The first two are analogous to theorems in Aumann/Shapley [1974]. The third is somewhat different because we allow the possibility the functions on which the value is defined can be determined by functionals that are 0 on the distinguished vector $x_0$, which corresponds in our theory to $I$ in the theory of Aumann and Shapley.

Admittedly, the Banach space setting is a highly abstract place to study the value of a game. On the other hand, various specific Banach spaces of functions are handy places to model games with infinitely many players. For instance, in the space $B(I, \mathcal{B})$ of bounded Borel measurable functions on the interval $I = [0,1]$ one can think of characteristic functions as denoting coalitions and functions $f$ with $0 \leq f(t) \leq 1$ for all $t$ as coalitions in which some members hold back their total resources. With a little imagination one can place interpretations on functions which are not bounded by 1 and even functions which are not always positive. The theory presented in this paper allows one to replace the space $B(I, \mathcal{B})$ with other spaces of functions which may be more appropriate to the problem being modelled or easier to handle. In section 5 we list several possible spaces including three Hilbert spaces and calculate the values of various games. The results in this paper also apply to spaces of sequences and allows the study of games with a countably infinite number of players.

1. Value and Spaces, Main Results

Let $E$ denote a Banach space with dual space $E'$. Let $X$ denote a linear space of real valued functions defined on $E$ which contains $E'$ as a subspace and has the property that whenever $F$ is in $X$ and $T$ is a continuous linear mapping from $E$ into $E$ the function $F \circ T$ is also in $X$. Here, $F \circ T$ is the function defined on $x$ in $E$ by $(F \circ T)(x) = F(Tx)$.

1.1 Definition. Given a distinguished vector $x_0$ in $E$, an $x_0$-value (on $X$) is defined to be a projection $P$ from $X$ onto $E'$ which satisfies the following two hypotheses:

(VA) $(PF)x_0 = F(x_0)$ for all $F$ in $X$.
(VB) If $T$ is a continuous isomorphism from $E$ into $E$ such that $Tx_0 = x_0$ then

$$P(F \circ T) = (PF) \circ T$$

for all $F$ in $X$.

Let $\mathcal{P}(E')$ denote the collection of all functions $F$ on $E$ which have the form

$$F(x) = p(f_1(x), f_2(x), \ldots, f_n(x))$$

where $p$ is a polynomial with $p(0, 0, \ldots, 0) = 0$ and $f_1, f_2, \ldots, f_n$ are in $E'$. It is not hard to see that if $F$ is in $\mathcal{P}(E')$ so is $F \circ T$ for any continuous linear mapping from $E$ into $E$. 