Abstract: Symmetric solutions (symmetric stable sets) and their uniqueness are investigated for symmetric games when the cores are large enough to have intersections with at least 4-dimensional surfaces of the imputation set. From this investigation, symmetric solutions for all five-person symmetric games with nonempty cores are obtained, as well as a sufficient condition for the uniqueness of symmetric solutions for symmetric games with nonempty cores.

1. Introduction

The purpose of this paper is to investigate symmetric solutions (symmetric stable sets) for symmetric games with nonempty cores, classifying such games according to the properties of the intersection of the core with the imputation simplex.

Let \((n, v)\) be a \((0,1)\)-normalized, \(n\)-person, symmetric game, i.e., \(v\) is a real-valued function (called a characteristic function) on \(\{0,1,\ldots,n\}\) with \(v(0) = v(1) = 0\) and \(v(n) = 1\), and \(n\) is the number of players. Let \(V\) denote a set of all such characteristic functions. In what follows, we might call a characteristic function \(v\) itself a game. We will use \(N\) to denote a set of all players. Let \(A\) be a set of all imputations, i.e.,

\[
A = \{x \in R^n | \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for all } i \in N\}
\]

where \(R^n\) is an \(n\)-dimensional Euclidean space. Note that \(A\) is an \((n-1)\)-dimensional unit simplex. For \(x, y \in A\) and nonempty \(S \subseteq N\), we say \(x\) dominates \(y\) via \(S\), denoted by \(x \dom_S y\), if \(x_i > y_i\) for all \(i \in S\) and \(\sum_{i \in S} x_i \leq v(|S|)\) where \(|S|\) denotes the cardinality of \(S\). The latter condition is called the effectiveness of \(S\) for \(x\). We write \(x \dom y\) if there is some \(S \subseteq X\) such that \(x \dom_S y\) via \(S\). For \(B \subseteq A\), let \(\Dom B = \{x \in A | \text{there is some } y \in B \text{ such that } y \dom x\}\). A set \(K \subseteq A\) is called a solution (a stable set) if (i) \(K \cap \Dom K = \emptyset\) (internal stability) and (ii) \(K \cup \Dom K = A\) (external stability). A set \(C \subseteq A\) is called the core if

\[
C = \{x \in A | \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\}.
\]

---

1) This research was supported in part by the Sakkokai Foundation, Japan. A part of this research was done while the author was a Visiting Research Associate at Cornell University in the summer 1981.

2) Prof. Shigeo Muto, Faculty of Economics, Tohoku University, Kawauchi Sendai 980, Japan.

0020-7276/83/040207-223$2.50 © 1983 Physica-Verlag, Vienna.
Since we are concerned with symmetric solutions throughout this paper, we now introduce some notation which will be useful for our discussions. First we note that a set $W \subseteq R^n$ is called symmetric if $x \in W$ implies that all $n$-dimensional vectors obtained from $x$ by permuting its coordinates are also contained in $W$. Let $R^n_s = \{x \in R^n | x_1 \leq x_2 \leq \ldots \leq x_n\}$, and for any $x \in R^n_s$, let $\pi(x)$ be a set of all $n$-dimensional vectors obtained from $x$ by permuting its coordinates. For any $W \subseteq R^n_s$, let $\pi(W) = \bigcup_{x \in W} \pi(x)$. For simplicity, denote $\sum_{i=s}^t x_i$ by $x(s, t)$ for any $x \in R^n_s$, and use $x(t)$ to denote $x(1, t)$. Let $A_s = \{x \in R^n_s | x(n) = 1, x_1 \geq 0\}$. Then we have $A = \pi(A_s)$. $A_s$ is called an ordered imputation set. For any $x, y \in A_s$ and nonempty $S = \{i(1), \ldots, i(s)\} \subseteq N$ with $i(1) \leq \ldots \leq i(s)$, we say $x$ dominates $y$ via $S$, denoted by $x \text{ dom}_S y$, if $x_{i(j)} > y_j$ for all $j = 1, \ldots, s$ and 

$$\sum_{j=1}^s x_{i(j)} \leq v(s)$$

We write $x \text{ dom}_S y$ if there is some $S \subseteq N$ such that $x \text{ dom}_S y$ via $S$.

For $B_s \subseteq A_s$, let $\text{ Dom}_s B_s = \{x \in A_s | \text{ there is some } y \in B_s \text{ such that } y \text{ dom}_s x\}$. Let $K_s$ be a subset of $A_s$ satisfying (i) $K_s \cap \text{ Dom}_s K_s = \emptyset$ and (ii) $K_s \cup \text{ Dom}_s K_s = A_s$. Then it is easily shown that $\pi(K_s)$ is a symmetric solution [for its detail, see Hart]. The core $C$ is given by $C = \pi(C_s)$ where $C_s = \{x \in A_s | x(s) \geq v(s) \text{ for all } s = 1, \ldots, n-1\}$. In what follows, our discussion will be proceeded exclusively on the ordered imputation set $A_s$, and thus, to simplify notation, we will eliminate $<$ and use $A, K, C, \text{ dom}_s, \text{ Dom}_s$ to denote $A_s, K_s, C_s, \text{ dom}_s, \text{ Dom}_s$.

Symmetric solutions have been determined for several classes of symmetric games, for example, three- and four-person games [von Neumann/Morgenstern; Nering], classes of extreme games [Griesmer; Rosenmüller], games reflecting real voting and economic situations [Bott; Hart; Muto], and so on. Their uniqueness have also been studied for some of these classes of games. In general, games which have no solution have been found by Lucas [1968] and Lucas/Rabie [1982]. These games are, however, nonsymmetric, and for symmetric games the existence of a solution (or a symmetric solution) is still unsolved in case where there are more than four players. Moreover, for most symmetric games with more than four players whose symmetric solutions have been determined, their cores are empty or very small. As to symmetric games which have large cores, a few works have been done for their symmetric solutions\(^3\), although the condition for the equivalence of the core and symmetric solutions was fully characterized by Shapley [1973] and Menshikova [1977]. Our object in this paper is to investigate symmetric solutions and their uniqueness for symmetric games when they have large cores. As a result, symmetric solutions for all five-person symmetric games with nonempty cores are obtained, as well as a sufficient condition for the uniqueness of symmetric solutions for symmetric games with nonempty cores.

\(^3\) For games with $v(s) = 0$ for all $s < n - 2$, symmetric solutions were determined by Lucas [1966] for all values of $v(n - 1)$, in a more general setup including nonsymmetric games.