A Degenerate Parabolic Equation
Modelling the Spread of an Epidemic (*).

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Summary. – We consider the Cauchy problem for a degenerate parabolic equation, not in divergence form, representing the diffusive approximation of a model for the spread of an epidemic in a closed population without remotion. We prove existence and uniqueness of the weak solution, defined in a suitable way, and some qualitative properties.

1. Introduction.

In this paper we shall consider the diffusive approximation (see [10]) of a model of the type proposed in [4], [7] for the spread of epidemic in a closed population without remotion.

The population is divided in susceptibles $s$ and infectious $i$, and $s + i = 1$ (after normalizing the total population to 1). The evolution law for $s$ is:

\[
\frac{\partial s(x, t)}{\partial t} = - s(x, t) \int K(x - y) i(y, t) \, dy
\]

where the convolution kernel is positive and with compact support.

We shall consider a one dimensional problem in the whole space. It can be found (see [3]) that (1.1), after suitable re-scaling, has the following diffusive approximation:

\[
s_t = ss_{xx} - s(1 - s) \quad \text{in } \mathbb{R} \times (0, T).
\]

This approximation is meaningful only when $s$ is sufficiently smooth and $0 < s < 1$. Moreover (see (1.1)) $s$ should be decreasing in time and such that the set $P(s) = \{ x : s(x, t) > 0 \}$ is constant in time.

If we want that a solution of (1.2) has these qualitative properties we have to impose some conditions on the initial datum $s(x, 0) = s_0(x)$. More precisely:

\[
(\text{HA}) \quad s_0 \in C^2(\mathbb{R}) , \quad 0 < s_0 < 1 , \quad s_0' - (1 - s_0) < 0 \quad \text{for } x \in \mathbb{R}.
\]

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The Cauchy problem for (1.2) is quite interesting since equation (1.2) is a non-linear parabolic equation degenerating in the points \((x, t)\) such that \(s(x, t) = 0\), and not in divergence form. Therefore we will study it with assumptions on the initial datum \(s_0\) less restrictive than (HA).

More precisely we will assume throughout the paper:

\((HB)\) \(s_0 \in C(\mathbb{R})\), \(s_0\) uniformly Lipschitz in \(\mathbb{R}\), (with Lipschitz constant \(M\)), \(0 < s_0 < 1\) for \(x \in \mathbb{R}\).

Since the equation (1.2) is a degenerate parabolic equation we cannot expect in general to have classical solutions. Because of the close apparent relation of (1.2) with a filtration equation with absorption (see [1]), the more natural definition of a weak solution of the Cauchy problem for equation (1.2) with initial datum \(s(x, 0) = s_0(x)\) (we will refer to this problem as Problem 1) seems to be the following:

**Definition 1.** A function \(s(x, t)\) is a weak solution to Problem 1 if:

1) \(0 < s < 1\), \(s \in C(\mathbb{R} \times [0, T])\), \(|s(x', t) - s(x'', t)| < M|x' - x''|\), \(\forall t > 0, x', x'' \in \mathbb{R}\), \(M\) positive constant;

2) \(s\) satisfies the following integral equation

\[
\frac{1}{\pi} \int \int \left[ -f(x, t) + s(x, t) \right] g(x, t) \, dx \, dt = 0
\]

for any \(f(x, t) \in F\), \(F = \{f \in C^1(\mathbb{R} \times [0, T])\}, f\) with compact support in \(x\), \(\forall t, f(x, T) = 0\};

3) whenever \(s\) is positive, it is a classical solution of equation (1.2).

Let us give some examples of explicit weak solutions (in the sense of Definition 1):

\[
\begin{align*}
s_1(x, t) &= \left( 1 - \cos \left( x - x_0 \right) \right) \left( 1 + \frac{2 - \frac{1}{c} \exp t}{c} \right)^{-1}, \quad 0 < \frac{1}{c} < 1 \\
s_2(x, t) &= \begin{cases} 
1, & |x - x_0| < 2\pi \\
0, & |x - x_0| > 2\pi
\end{cases} \\
s_3(x, t) &= |1 + A \cos (x - x_0)| \left( 1 + \frac{1 + A - \frac{1}{c} \exp t}{c} \right)^{-1}, \quad A > 1, \ 0 < \frac{1}{c} < 1 \\
s_4(x, t) &= \begin{cases} 
\sec \left( x - x_0 - \pi \right) < \frac{1}{A} \cos \frac{1}{A}, & \pi \cos \frac{1}{A} \\
0, & |x - x_0 - \pi| \geq \frac{1}{A} \cos \frac{1}{A}
\end{cases}
\end{align*}
\]