A Note on the Convergence of Transfer Sequences in $n$-Person Games

By G. Owen, Houston

Abstract: It was shown by Stearns [1968], and more recently by Kalai, Maschler, and Owen [1973], that a certain iterative technique converges to points in the bargaining sets. We prove in this paper that the conditions on the iterative technique can be relaxed to a certain extent. A counter-example shows that no further relaxation can be allowed.

Let $(N, v)$ be an $n$-person game in characteristic function form. For a payoff vector $x = (x_1, \ldots, x_n)$ and a coalition $S$, we define the excess of $S$ in $x$ by:

$$e(S, x) = v(S) - \sum_{i \in S} x_i$$

(1)

Let $S_1, S_2, \ldots, S_{2^n}$ be the subsets of $N$, ordered by

$$e(S_k, x) \geq e(S_{k+1}, x) \quad k = 1, 2, \ldots, 2^n - 1.$$  

(2)

Define the vector

$$\theta(x) = (e(S_1, x), e(S_2, x), \ldots, e(S_{2^n}, x)).$$

(3)

The surplus of $i$ against $j$ in $x$ is defined by

$$s_{ij}(x) = \max e(T, x)$$

(4)

where the maximum is taken over all $T$ such that $i \in T$, but $j \notin T$.

We shall let $X$ be a convex subset of $R^n$. In particular, $X$ may be $R^n$ itself, or it may be the set of all imputations; more generally, it may be chosen as a convex polyhedral set.

In Kalai, Maschler, and Owen [1973], a demand function was defined as a set of $n^2$ lower semicontinuous functions $d_{ij}(x)$, $i, j = 1, \ldots, n$, satisfying

$$0 \leq d_{ij}(x) \leq \frac{1}{2} (s_{ij}(x) - s_{ji}(x)) \quad \text{if} \quad s_{ij} > s_{ji}$$

(5)
\[ d_{ij}(x) = 0 \quad \text{if} \quad s_{ij} \leq s_{ji} \quad \text{or if} \quad i = j. \quad (6) \]

For any \( x \in X \), and any pair \( i, j \), let \( x' \) be defined by

\[
x'_k = \begin{cases} 
  x_k & \text{if} \ k \neq i, j \\
  x_i + d_{ij}(x) & \text{if} \ k = i \\
  x_j - d_{ij}(x) & \text{if} \ k = j
\end{cases} \quad (7)
\]

then \( x' \in X \).

The bargaining set \( M_D(x) \) is the set of all vectors \( x \in X \) such that \( d_{ij}(x) = 0 \) for all \( i, j \).

Define a sequence of vectors \( x^0, x^1, x^2, \ldots \), as follows: let \( x^0 \in X \) be arbitrary. Then, given \( x^t \), choose \( i(t), j(t) \) satisfying

\[
d_{i(t)j(t)}(x^t) = \max_{i, j} d_{ij}(x^t) \quad (8)
\]

and use these to define \( x^{t+1} \) by

\[
x^{t+1}_{i(t)} = x^t_{i(t)} + d_{i(t)j(t)}(x^t) \quad (9)
\]

\[
x^{t+1}_{j(t)} = x^t_{j(t)} - d_{i(t)j(t)}(x^t) \quad (10)
\]

\[
x^{t+1}_m = x^t_m \quad \text{for all other} \ m. \quad (11)
\]

It is shown in Kalai, Maschler, and Owen [1973] that this sequence always converges, and that the limit \( x^* \) will always belong to \( M_D(X) \).

We shall prove below that, if condition (5) is replaced by the weaker condition

\[
0 \leq d_{ij}(x) \leq \alpha (s_{ij}(x) - s_{ji}(x)) \quad \text{if} \quad s_{ij} > s_{ji} \quad (12)
\]

where \( \alpha \) is a constant, \( 0 < \alpha < 1 \), then the sequence will still converge to a point in the set \( M_D(X) \).

The following lemma is necessary:

**Lemma:** For \( 0 < \beta < 1 \), and any \( m \)-vector \( y = (y_1, \ldots, y_m) \), define

\[
G(y) = \sum_{k=1}^{m} \beta^k y_k. \quad (13)
\]

Suppose that \( G(y) > 0 \). Then there exists some \( k \) such that \( y_k > 0 \) and, moreover,

\[
y_k > -\frac{1 - \beta}{\beta} \sum_{l=1}^{k-1} y_l \quad (14)
\]