Semi-Symmetric Solutions for \((n, k)\) Games\(^1\)

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Abstract: The concept of semi-symmetric solution for \((n, k)\) games was presented by Muto [1978]. An extension of this type of solution is studied here.

1. Introduction

The purpose of this paper is to extend known results on semi-symmetric solutions for \((n, k)\) games.

Let \((n, v)\) be a \((0, 1)\)-normalized, \(n\)-person, symmetric game, i.e., \(v\) is a real-valued function, called a characteristic function, on \(\{0, 1, \ldots, n\}\) with \(v(0) = v(1) = 0\) and \(v(n) = 1\), and \(n\) is the number of players. We will use \(N\) to denote the set of players. The game \((n, v)\) is an \((n, k)\) game if \(v(s) = 0\) for all \(s < k\), \(0 < v(k) \leq 1\), and \(v(s) \leq (s/k) \cdot v(k)\) for all \(k < s < n\). In what follows we will deal with the two types of \((n, k)\) games, i.e., (i) \(n = qk + \gamma\) where \(q\) and \(\gamma\) are integers satisfying \(q \geq 2\) and \(0 \leq \gamma \leq k - 1\), and (ii) \(n = 2k - 1\).

Let \(A\) be the set of imputations, i.e., \(A = \{x \in \mathbb{R}^N \mid \sum_{i=1}^{N} x_i = 1, x_i \geq 0 \text{ for all } i \in N\}\) where \(\mathbb{R}^N\) denotes the \(n\)-dimensional Euclidean space. For \(x = (x_1, x_2, \ldots, x_n) \in A\) and for any \(1 \leq m \leq n\), \((x_1, \ldots, x_m, |x_{m+1}, \ldots, x_n|)\) denotes the set of imputations obtained from \(x\) by permuting its coordinates \(x_1, \ldots, x_m\). For \(x, y \in A\) and nonempty \(S \subseteq N\), we say that \(x\) dominates \(y\) via \(S\), denoted by \(x\) dom \(y\) via \(S\), if \(x_i > y_i\) for all \(i \in S\), and \(\sum_{i \in S} x_i \leq v(|S|)\) where \(|S|\) denotes the cardinality of \(S\). The latter condition is called the effectiveness of \(S\) for \(x\). We write \(x\) dom \(y\) if there is some \(S \subseteq N\) such that \(x\) dom \(y\) via \(S\). For \(B \subseteq A\), let \(\text{Dom } B = \{x \in A \mid \text{there is some } y \in B \text{ such that } y\text{ dom } x\}\). A set \(K \subseteq A\) is a solution (or a stable set) if it satisfies the two conditions, (i) \(K \cap \text{Dom } K = \emptyset\) (internal stability), and (ii) \(K \cup \text{Dom } K = A\) (external stability).

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Let \( \omega = v(k)/k \), and we will define semi-symmetric sets for \((n, k)\) games. First we consider \((n, k)\) games with \( n = qk + \gamma \) where \( q \) and \( \gamma \) are integers satisfying \( q \geq 2 \) and \( 0 \leq \gamma < k \). Let \( S = \{S_1, \ldots, S_q, S_{q+1}\} \) be a partition of \( N \) where \( |S_j| = k \) for all \( j = 1, \ldots, q \) and \( |S_{q+1}| = \gamma \). If \( S_{q+1} \neq \emptyset \), then define \( S'_{q+1} = S_{q+1} \cup T \) where

\[
|S'_{q+1}| = k \quad \text{and} \quad T = \bigcap_{j=1}^{q} T_j \quad (T_j \subseteq S_j).
\]

Assume \((t-1) \cdot v(k) < 1 < t \cdot v(k)\) for some \( t = 1, \ldots, q + 1 \). Let \( \Omega = tk\omega \) if \( t < q \), and let \( \Omega = (qk + \gamma) \omega \) if \( t = q + 1 \). A set \( K_{S, \omega} (n, k) \subseteq A \) is a semi-symmetric set for \((n, k)\) games with \( n = qk + \gamma \) \((q \geq 2, 0 \leq \gamma < k \leq 1)\) with respect to \( S \) and \( \omega \) if

\[
K_{S, \omega} (n, k) = \bigcup_{j=1}^{t} K_j
\]

where

\[
K_j = \{x \in A \mid \sum_{i \in S_j} x_i = v(k) - \Omega; x_i \geq \omega \text{ for at least one } i \in S_j; \quad x_i = \omega \text{ for all } i \in \bigcup_{l=1}^{t} S_l - S_j; \quad x_i = 0 \text{ for all } i \in N - \bigcup_{l=1}^{t} S_l \} \quad \text{for } j = 1, \ldots, \min(t, q),
\]

and if \( t = q + 1 \), then

\[
K_{q+1} = \{x \in A \mid \sum_{i \in S_{q+1}} x_i = v(k) - \Omega; x_i \geq \omega \text{ for all } i \in T; \quad x_i = \omega \text{ for all } i \in S'_{q+1} \}.
\]

For \((n, k)\) games with \( n = 2k - 1 \), let \( S = \{S_1, S_2, i(0)\} \) be a partition of \( N \) where \( |S_1| = |S_2| = k - 1 \) and let \( \Omega = n \cdot \omega - 1 \). A set \( K_{S, \omega} (n, (n + 1)/2) \) is a semi-symmetric set for \((n, k)\) games with \( n = 2k - 1 \) with respect to \( S \) and \( \omega \) if

\[
K_{S, \omega} (n, (n + 1)/2) = \{x \in A \mid \sum_{i \in S_1 \cup \{i(0)\}} x_i = 1 \} \text{ when } v(k) = 1
\]

and when \( v(k) < 1 \), \( K_{S, \omega} (n, (n + 1)/2) = K_1 \cup K_2 \)

where

\[
K_1 = \{x \in A \mid \sum_{i \in S_1 \cup \{i(0)\}} x_i = v(k) - \Omega; \quad x_i = \omega \text{ for at least one } i \in S_1 \cup \{i(0)\}; \quad x_i = \omega \text{ for all } i \in S_2 \}
\]

and

\[
K_2 = \{x \in A \mid \sum_{i \in S_2 \cup \{i(0)\}} x_i = v(k) - \Omega; \quad x_{i(0)} = \omega; \quad x_i = 0 \text{ for all } i \in S_1 \}.
\]

The following results were proved in Muto [1978].

**Theorem 1:** Consider \((n, k)\) games with \( n = qk + \gamma \) \((q \geq 2, 0 \leq \gamma < k - 1)\).

a) If \( v(k) = 1, 1/2, \ldots, 1/(q - 1) \) or \( 1/q \), then \( K_{S, \omega} (n, k) \) is a solution.

b) If \( v(k) = k/n \) and \( \gamma > 0 \), then \( K_{S, \omega} (n, k) \) is a solution if and only if we can take \( T_j \)'s so that \( |T_j| \leq 1 \) for all \( j = 1, \ldots, q \).