Dynamic Systems of Differential Inclusions for the Bargaining Sets

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Abstract: Dynamic systems of differential inclusions, leading to the appropriate bargaining sets, are introduced. Stability properties of these systems are studied.

1. Introduction

Maschler/Peleg [1976] investigated dynamic systems of the form

\[ x^{t+1} \in \varphi (x^t), \quad t = 0, 1, 2, \ldots \]

where \( \varphi \) is a set-valued function. In this paper we are concerned with analogous systems of the form

\[ \dot{x}(t) \in F(x(t)), \quad (1.1) \]

where \( F(x) = \varphi (x) - x \). Analogous results to the main results of Maschler/Peleg [1976, Sections 3 and 4] hold in the present case. These results are generalized in this paper.

The theoretical base of our investigation is provided in Section 2. Sufficient and necessary conditions, for a given set to be a stable set, are given. In Section 3 we introduce dynamic systems of the form (1.1) having the following property: For any initial point \( x \), there exist solutions of these systems that start at \( x \) and converge to the appropriate bargaining set of a given game. These systems include, as particular cases, Billera's [1972] systems and analogous of Stearns' [1968] schemes. In Section 4 we deduce the stability of Schmeidler's [1969] nucleolus, each point of the lexicographic kernel and every nonempty strong \( e \)-core with respect to each of these systems.

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2. Stable Sets and Stable Points

Let $X$ be a closed subset of $\mathbb{R}^n$ and let $F$ be a multifunction (namely, set-valued function) from $X$ into $\mathbb{R}^n$. We consider the following differential inclusion:

$$\dot{x}(t) \in F(x(t)).$$

(2.1)

A solution of (2.1) is any absolutely continuous function $x : \mathbb{R}_+ \rightarrow X$ that satisfies (2.1) for almost all $t$ in $\mathbb{R}_+$.

The definitions that will be introduced in this section are analogous to appropriate definitions of Maschler/Peleg [1976] with respect to discrete systems.

A point $x$ in $X$ is called a critical point of $F$ if

$$F(x) = \{0\}.$$  

(2.2)

**Definition 2.1:** A nonempty subset $Q$ of $X$ is called stable w.r.t. (with respect to) $F$, if for every neighborhood $U$ of $Q$ there exists a neighborhood $V$ of $Q$ such that, for any solution $x$ of (2.1), $x(0) \in V$ implies $x(t) \in U$ for all $t$ in $\mathbb{R}_+$.

**Remark 2.2:** $X$ is always a stable set.

**Definition 2.3:** A point $x$ in $X$ is stable if $\{x\}$ is stable.

**Definition 2.4:** Let $g : X \rightarrow \mathbb{R}$ and let $\alpha \in \mathbb{R}$. $g$ is called an $\alpha$-Lyapunov function for $F$ if for any solution $x$ of (2.1),

$$0 \leq s \leq t \Rightarrow g(x(t)) - g(x(s)) \leq \alpha \| x(t) - x(s) \|,$$

(2.3)

where $\| \cdot \|$ is some fixed norm in $\mathbb{R}^n$.

**Proposition 2.5:** Suppose that a Lipschitz function $g : X \rightarrow \mathbb{R}$ satisfies:

$$x \in X \text{ and } y \in F(x) \Rightarrow g'(x; y) \leq \alpha \| y \|,$$

(2.4)

for some $\alpha \leq 0$, where

$$g'(x; y) = \liminf_{r \to 0^+} \frac{g(x + ry) - g(x)}{r}.$$  

(2.5)

Then, $g$ is $\alpha$-Lyapunov.

See, e.g., Yarom [1983, Th. 2.4] for the proof of Proposition 2.5.

**Definition 2.6:** Let $g : X \rightarrow \mathbb{R}^m$. $g$ is called $F$-monotone if each of its components is 0-Lyapunov.

We quote the following two definitions of Maschler/Peleg [1976].