On the Stationary, Compressible and Incompressible Navier-Stokes Equations (*)

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Summary. In this paper we study the system (1.1), (1.3) which describes the stationary motion of a given amount of a compressible heat conducting, viscous fluid in a bounded domain $\Omega$ of $\mathbb{R}^n$, $n > 2$, and we consider the incompressible limit of the solutions of that system of equations (for barotropic flows) as the Mach number becomes small.

1. Introduction.

In this paper we study the system

\begin{equation}
\begin{aligned}
- \mu \Delta u - \nu \nabla \text{div} u + \nabla p(q, \zeta) &= q(f - (u \cdot \nabla)u), \quad \text{div}(qu) = g, \\
- \chi \Delta \zeta + c_r q u \cdot \nabla \zeta + \zeta p'_r(q, \zeta) \text{div} u &= \phi h + \psi(u, u), \quad \text{in } \Omega, \\
u|_\Gamma = 0, \quad \zeta|_\Gamma = \zeta_0,
\end{aligned}
\end{equation}

(1.1)

in a bounded open domain $\Omega$ in $\mathbb{R}^n$, for arbitrarily large $n > 2$. It is assumed that $\Omega$ lies (locally) on one side of its boundary $\Gamma$, a $C^{r,s}$ manifold. Here,

\begin{equation}
\psi(u, u) = \chi \sum_{i=1}^{n} \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} \right)^2 + \chi_1 (\text{div } u)^2,
\end{equation}

(1.2)

and $(v \cdot \nabla)u = \sum_{i=1}^{n} v_i (\partial u_i/\partial x_i)$. System (1.1) describes the stationary motion of a compressible, heat conductive, viscous fluid (see Serrin [9]).

In equation (1.1), $u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$ is the velocity field, $q(x)$ is the density of the fluid, $\zeta(x)$ is the absolute temperature, $f(x)$ and $h(x)$ are the assigned external force field and heat sources per unit mass, and $p(q, \zeta)$ is the pressure.

In the physically significant case one has $g = 0$, however, it is not without interest, from a mathematical point of view, to study the general case.

In order to avoid technicalities, we will assume that the coefficients $\mu > 0$, $\nu > - \mu$, $\chi > 0$, $c_r$, $\chi_0$, $\chi_1$, are constant and that $\zeta_0 > 0$ is constant.

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Since the total mass of fluid is given, we impose the condition

\[ \frac{1}{|Q|} \int_Q \rho(x) \, dx = m, \quad \text{or equivalently} \quad \frac{1}{|Q|} \int_Q \sigma(x) \, dx = 0, \]

where \( m > 0 \) is given, and \( \rho(x) \) is defined by setting \( \rho(x) = m + \sigma(x) \).

The function \( p(q, \zeta) \) is defined, and

\[ p(q, \zeta) \in C^{+\infty}([m-l, m+l] \times [\zeta_0 - l_1, \zeta_0 + l_1]), \]

where \( 0 < l < m/2, 0 < l_1 < \zeta_0/2. \)

Consequently, we can write

\[ p^i(m, q, \zeta_0 + x) = k + \omega_1(q, x), \]
\[ p^j(m, q, \zeta_0 + x) = \omega_2(q, x), \]

where \( k = p^j(m, q, \zeta_0), \omega_1(0, 0) = 0, \) and \( \omega_1, \omega_2 \) are in the space \( C^{+\infty}(I(l, l_1)) \), where \( I(l, l_1) = [-l, l] \times [-l_1, l_1]. \) We assume that \( k > 0 \) (in fact, \( k \neq 0 \) would be sufficient here).

By setting

\[ q = m + \sigma, \quad \zeta = \zeta_0 + x, \]

we write system (1.1) in the equivalent form

\[
\begin{cases}
- \mu \Delta u - \nu \nabla u + k \nabla \sigma = F(f, u, \sigma, z), \\
m \nabla u + u \cdot \nabla \sigma + \sigma \nabla u = g, \\
- \chi \Delta z = H(h, u, \sigma, z), \quad \text{in } \Omega, \\
u|_{\partial \Gamma} = 0, \quad z|_{\partial \Gamma} = 0,
\end{cases}
\]

where by definition,

\[
\begin{cases}
F(f, u, \sigma, z) = (\sigma + m)(f - (u \cdot \nabla) u) - \omega_1(\sigma, z) \nabla \sigma - \omega_2(\sigma, z) \nabla z, \\
H(h, u, \sigma, z) = (\sigma + m)h - c_0(m + \sigma)u \cdot \nabla z + \varphi(u, u) + \frac{\zeta_0 + x}{m + \sigma} \omega_2(\sigma, z)(u \cdot \nabla \sigma - g).
\end{cases}
\]

Note that equation (1.5) is used in deducing the expression \( H. \)

In section 3, we prove the following result:

**Theorem A.** Let \( p \in [1, +\infty[ \) and \( j > -1 \) verify \((j+2)p > n. \) Assume that \( F \in C^{+\infty} \) and \( p(q, \zeta) \in C^{+\infty}(I(l, l_1)). \) If \( (j, g, h) \in W^{j+1,p} \times W^{j+2,p} \times W^{-1,p} \) verify the