Regularly Hyperbolic Systems and Gevrey Classes (*).

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Sunto. – In questo lavoro mostriamo che il problema di Cauchy per il sistema del 1° ordine alle derivate parziali

\[
I[U] = U_t - \sum_{k=1}^{n} A_k(t, x) U_{x_k} - B(t, x) U = 0 \quad \text{on } \mathbb{R}^n \times [0, T]
\]

sotto le ipotesi che l'operatore \( L \) sia regolarmente iperbolico, che i coefficienti siano nella classe di Gevrey \( \gamma^{(1)} \) in \( x \) e che i coefficienti \( A_k \) della parte principale siano holderiani di ordine \( \alpha \) in \( t \), è ben posto in \( \gamma^{(1)} \), a condizione che risulti

\[
1 < s < \frac{1}{1 - \alpha}.
\]

1. – Introduction.

Let us consider the following Cauchy problem

\[
\begin{aligned}
U_t &= \sum_{k=1}^{n} A_k(t, x) U_{x_k} + B(t, x) U \\
U(0, x) &= \varphi(x)
\end{aligned}
\quad \text{on } \mathbb{R}^n \times [0, T]
\]

where \( A \) and \( B \) are \( N \times N \) real matrices, while \( U \) and \( \varphi \) are real \( N \)-vectors.

We shall assume that system (1) is regularly hyperbolic, i.e. the equation

\[
\det \left( \lambda I - \sum_{k=1}^{n} A_k(t, x) \xi^{(k)} \right) = 0
\]

has, for any \((x, \xi) \in \mathbb{R}^{2n}\), for any \( t \in [0, T] \), \( N \) real and distinct roots; moreover, the coefficients \( A_k(t, x) \) are bounded on \( \mathbb{R}^n \times [0, T] \).

If \( X \) is a space of functions or functionals on \( \mathbb{R}^n \), we shall say that problem (1) is \( X \)-well—posed if, for any \( \varphi \) in \( X \), it admits one and only one solution \( U \) in \( C([0, T]; X) \).

It's a well known fact that, if the coefficients \( A_k \) are lipschitz-continuous in \( t \) and \( C^\infty \) in \( x \) and if \( B \) is \( C^\infty \) in \( x \), then problem (1), under the hypothesis of regular hyperbolicity, is \( C^\infty \)-well—posed (see, for instance, [4]).

(*) Entrato in Redazione il 10 aprile 1984.

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On the other hand, if the coefficients $A_\xi$ are only Hölder-continuous in $t$, problem (1) is not (in general) $C^{\alpha}$-well-posed: this situation was studied for the first time by F. Colombari, E. De Giorgi and S. Spagnolo, who have proved in [1] that the Cauchy problem for a second order hyperbolic equation, with coefficients depending only on $t$ and Hölder-continuous of order $\alpha$, is well-posed in the Gevrey classes $\gamma^{(\alpha)}_{\text{loc}}$, provided that $1 < s < 1/(1 - \alpha)$. Their results have been recently extended by T. Nishitani in [5] to the case of a second order hyperbolic equation with coefficients Hölder-continuous in $t$ and Gevrey in $x$.

In the present work, we'll prove the following

**Theorem 1.** Let us consider problem (1) under the hypothesis of regular hyperbolicity. Let us suppose that:

i) the coefficients $A_{\xi}(t, x)$ are Hölder-continuous of order $\alpha$ in $t$ (uniformly with respect to $x$), where $0 < \alpha < 1$; moreover, they belong to the Gevrey class $\gamma^{(\alpha)}_{\text{loc}}$ in $x$;

ii) the coefficient $B(t, x)$ is locally bounded on $R^*_x \times [0, T]$ and belongs to the Gevrey class $\gamma^{(\alpha)}_{\text{loc}}$ in $x$.

Then problem (1) is $\gamma^{(\alpha)}_{\text{loc}}$-well-posed, provided that

$$1 < s < \frac{1}{1 - \alpha}.$$ 

Using the same techniques here employed in the proof of theorem 1, one can also prove the following

**Theorem 2.** Let us consider the Cauchy problem

$$A_{\xi}(t, x) U_x = \sum_{k=1}^{n} A_k(t, x) U_x + B(t, x) U \quad \text{on } R^*_x \times [0, T]$$

$$U(0, x) = \varphi(x)$$

under the following hypotheses:

i) (hyperbolicity) the matrices $A_i(t, x)$ ($i = 0, 1, \ldots, n$) are hermitian; moreover, the matrix $A_0(t, x)$ is strictly positive defined and the matrices $A_i(t, x)$ are bounded on $R^*_x \times [0, T]$ for $1 < h < n$, while the matrix $B(t, x)$ is locally bounded;

ii) all the coefficients belong to the Gevrey class $\gamma^{(\alpha)}_{\text{loc}}$ in $x$;

iii) the matrix $A_0(t, x)$ is Hölder-continuous of order $\alpha$ with respect to $t$ (uniformly with respect to $x$).