On Maximum Tests for Normal Distributions

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Abstract: Let \( y \) be a normally distributed random vector with known regular covariance matrix and let \( A, B \) be disjoint closed convex sets in \( \mathbb{R}^n \). To be tested is the zero-hypothesis \( E(y) \in A \) against the alternative hypothesis \( E(y) \in B \) at a level of significance \( \alpha \). Taking the set of admissible tests as one strategy set, the set of probability densities corresponding to \( B \) as the other strategy set and the power function of the test problem as the pay-off function this game has an equilibrium point. Thus there is a test, in particular a Neyman-Pearson test, which is simultaneously a maximin and a minimax test. The optimal test is uniquely determined, except on sets with measure zero. Finally the case of non-convex \( A, B \) is briefly considered.

1. Introduction

Let \( y \) denote a normally distributed random vector with known covariance matrix \( \Sigma, \Sigma \) regular. Let \( A, B \subseteq \mathbb{R}^n \) be non-void disjoint closed convex sets. In order to make the matter here not too complicated we additionally claim: if \( \{a_v - b_v\}_v \not\subset A - B \) is bounded \( \forall_v \), then both \( \{a_v\}_v \in A, \{b_v\}_v \in B \) are. We want to test the zero-hypothesis

\[ H_0: E(y) \in A \]

\((E: \text{expectation value})\) against the alternative hypothesis

\[ H_1: E(y) \in B \]

with level of significance \( \alpha, 0 < \alpha < 1 \); i.e. calling any Lebesgue-measurable function \( \psi: \mathbb{R}^n \to [0, 1] \) a test and denoting for

\[ \gamma = (\gamma_1, \ldots, \gamma_n)^T, \quad x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \]

\[ f_\gamma(x) := \det (\Sigma^{-1})^{1/2} (2\pi)^{n/2} \exp \left[ - \frac{(x - \gamma)^T \Sigma^{-1} (x - \gamma)}{2} \right] \]

and \( F_A := \{f_\gamma, \gamma \in A\}, F_B := \{f_\gamma, \gamma \in B\} \) we consider the set of tests

\[ \psi_\alpha := \{\psi, E(\psi, f_\gamma) \leq \alpha \forall \gamma \in A\}, E(\psi, f) := \int_{\mathbb{R}^n} \psi(x) f(x) \, dx. \]

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The expression $F(\psi, f), \psi \in \psi_\alpha, f \in F_A \cup F_B$ is called the power function of the test problem.

Generally there does not exist a uniformly most powerful test [Schmetterer]. In this case one often will use a maximin test, i.e. a test $\psi_0 \in \psi_\alpha$ with

$$\inf_{\gamma \in B} E(\psi_0, f_\gamma) = \sup_{\psi \in \psi_\alpha} \inf_{\gamma \in B} E(\psi, f_\gamma) \text{ [ibd. p. 178].}$$

Let us call $\psi' \in \psi_\alpha$ a minimax test if there is a $\gamma' \in B$ with

$$E(\psi', f_{\gamma'}) = \sup_{\psi \in \psi_\alpha} E(\psi, f_{\gamma'}) = \inf_{\gamma \in B} \sup_{\psi \in \psi_\alpha} E(\psi, f_\gamma).$$

2. Existence of a Saddlepoint

It is clear that if $(\psi^*, f_{\gamma^*})$ is a saddlepoint of the power function $E(\psi, f)$ on $\psi_\alpha \times F_B$, i.e. when for all $\psi \in \psi_\alpha, f_\gamma \in F_B$ the following double inequality holds

$$E(\psi, f_{\gamma^*}) \leq E(\psi^*, f_{\gamma^*}) \leq E(\psi^*, f_{\gamma^*}),$$

then $\psi^*$ is simultaneously a maximin and a minimax test.

We shall show that existence of a saddlepoint.

Let $\gamma' \in A, \gamma'' \in B$. The well-known Neyman-Pearson lemma [Schmetterer] yields

$$E(\psi, f_{\gamma'}) \geq E(\psi, f_{\gamma''})$$

for all test $\psi$ with $E(\psi, f_{\gamma'}) \leq 0$ for

$$\psi_{\gamma', \gamma''}(x) =
\begin{cases}
1 & (\gamma'' - \gamma')^T \Sigma^{-1} x \geq (\gamma'' - \gamma')^T \Sigma^{-1} \gamma' + [(\gamma'' - \gamma')^T \Sigma^{-1} (\gamma'' - \gamma')]^{1/2} \cdot U(1 - \alpha) \\
0 & \text{elsewhere}
\end{cases}$$

where $U$ denotes the inverse function of $\Phi(z) = \int_{-\infty}^{\frac{z}{\sqrt{2}}} e^{-t^2/2} dt / \sqrt{2\pi}$. Because

$A \cap B = \emptyset$, equality only holds if $\psi(x) = \psi_{\gamma', \gamma''}(x)$ almost everywhere. [ibd. p. 167].

We call $\psi_{\gamma', \gamma''}$ a Neyman-Pearson test when $\gamma' \in A, \gamma'' \in B$. One immediately sees that for an arbitrary $\gamma \in A \cup B$

$$E(\psi_{\gamma', \gamma''}, f_{\gamma'}) = \Phi((\gamma'' - \gamma')^T \Sigma^{-1} (\gamma - \gamma')/[(\gamma'' - \gamma')^T \Sigma^{-1} (\gamma'' - \gamma')]^{1/2} - U(1 - \alpha)).$$

hence especially

$$E(\psi_{\gamma', \gamma''}, f_{\gamma''}) = \Phi([\gamma'' - \gamma')^T \Sigma^{-1} (\gamma'' - \gamma')]^{1/2} - U(1 - \alpha).$$