NONLINEAR FREQUENCY SHIFT OF STATIONARY WAVES
IN AN ELECTRON BEAM

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The stationary oscillations of an electron beam have been repeatedly discussed [1-4]. Here, however, we single out the class of waves whose frequency does not depend on their amplitude. We show that under certain conditions a nonlinear shift in the frequency of a stationary wave is possible.

We write the original equations in the form [1]

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial n v}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} &= -\frac{e}{m} E, \\
\frac{\partial E}{\partial x} &= -4\pi e (a - N_0).
\end{align*}
\]

Assuming that all variables depend only on \(\xi = \omega t - kx\), we obtain from the equation of continuity the relation

\[
(a - kv) n = N_0,\quad p = \text{const.}
\]

where \(\omega_0 = \sqrt{4\pi e^2 N_0 / m}\), and \(p = \text{const.}\). In the dimensionless variables \(a = (kv - w) / \omega_0\), \(\psi = e k E / m \omega_0^2\), the equations of stationary oscillations take the form

\[
a'_1 = \psi/a, \quad \psi'_1 = -\left(\frac{E}{a} + 1\right).
\]

The phase plane of the system (3), constructed using the integral \((a + p)^2 + \psi^2 = c^2\), is shown in Fig. 1. In what follows, we restrict ourselves to a consideration of periodic waves, in which \(\pi < \infty (c < |p|)\). In such waves the amplitude of the field oscillations is \(\Delta E = l/2(E_{\text{max}} - E_{\text{min}}) = (m \omega_0^2 / k e)\); the particle velocity is greater than the phase velocity for \(p < 0\) (slow waves) and less than the phase velocity for \(p > 0\) (rapid waves) [1, 4].

We demand, as usual, that the period of the wave in the variable \(\xi\) is equal to \(2\pi\), defining \(\omega\) and \(k\) as the frequency and the wave number:

\[
\int_{-p+e}^{p+e} \frac{da}{\sqrt{a^2 - (a + p)^2}} = \pi
\]

(here and in what follows, the first sign refers to slow waves and the second to rapid waves). For a given \(k\), having applied constraints which remove the arbitrariness in the constants, we can construct a class of stationary waves, which contain only the amplitude and the phase of the wave as arbitrary parameters. Let us consider this using specific examples.

a) Consider the set of stationary waves which propagate under conditions of constant ion background density \(N_0\), and which are characterized by an average mass flux density \((\mathbf{n}) = \text{const} = N_0 v_0\) (here \(\langle \cdots \rangle\) indicates an average over the period, \(v_0\) is the particle velocity in an unperturbed beam, and, moreover, the fact that \(\langle n \rangle = N_0\) has been used). According to Eq. (4), \(p = -1\); then from Eq. (2) there follows the well-known result: \(\omega = kv_0 = \omega_0\) [4]. In this case, the frequency of the wave is independent of its amplitude.

b) Now permit the parameters of a stationary wave with wave number \(k\) to vary slowly with time due to the action of external sources in resonance with the wave, which we take into account by adding small terms
\( f_i(\xi, t) \) to the right-hand side of Eq. (1). In this way, we can take into account the excitation of stationary waves in the beam upon nonlinear resonance interaction of the waves (when the functions \( f_i \) describe the combination rf force resonant with the nonlinear wave), or their excitation by an external resonant beam. We construct the set of stationary solutions which permit us to describe the given quasistatic process.

Averaging Eq. (1) with external sources \( f_i \) over the spatial period, and using the neutrality of the beam on the average \( \langle E \rangle = 0 \), we obtain the following conservation laws:

\[
\langle n \rangle = \text{const} = \bar{N}_0, \quad \langle \psi \rangle = \text{const} = \bar{\psi}_0. \tag{5}
\]

The set of stationary solutions \( v(\xi), n(\xi) \), which contain average and oscillatory components, must be constructed in such a way that the conditions (5) are satisfied; the average velocity can easily be determined from Eq. (3):

\[
\langle \psi \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} -\frac{a^2 \sin \theta}{\sqrt{\psi^2 - (a + \rho)^2}} \, \sin \phi \, d\phi = \pm \left( \frac{\rho^2}{2} + \rho^3 \right). \tag{6}
\]

Also using the fact that \( \langle n \rangle = \bar{N}_0 \) (i.e., \( \bar{N}_0 = \bar{N}_0 \)), we obtain from Eq. (5) the dispersion relation in explicit form:

\[
\omega = k v_0 - \rho (1 + \frac{1}{2} \psi^2). \tag{7}
\]

In this class of stationary waves, the dependence of frequency on amplitude is explained by the presence of an average drift speed of the individual oscillators, which is different for waves of different amplitude.

c) Now consider an adiabatic spatial variation of a stationary wave with frequency \( \omega \) under the action of external sources \( f_i(\xi, x) \). Using the condition \( \langle E \rangle = 0 \), we obtain the following conservation laws:

\[
\langle \psi \rangle = \psi_0^2, \quad \langle n \psi \rangle = \bar{N}_0 \psi_0. \tag{8}
\]

where \( \bar{N}_0 \) and \( \psi_0 \) are the density and particle velocity of the unperturbed beam. Since \( \langle n \rangle = \bar{N}_0, p = \pm 1 \), and \( \langle \psi^2 \rangle = \frac{3}{2} \psi_0^2 + 1 \), we find from Eq. (8) that

\[
\left( \frac{3}{2} \pm \frac{\psi_0}{\omega_p} \right) \psi_0^2 e^t - k \psi_0^2 - (\omega_p \pm \omega)^2, \tag{9}
\]

\[
\omega \pm \omega_p = kv_0 \omega_p^2 / \omega_p^2 \quad (\omega_p \equiv \omega_p(\bar{N}_0)).
\]

Equations (9) determine the dependence of \( k \) and \( \bar{N}_0 \) on frequency and amplitude [in the presence of sources \( f_i(\xi, x) \), which determine the profile \( c(x) \), the corresponding ion density distribution arises as a result of the condition of neutrality of the system]. The dispersion relation can be written explicitly when \( \omega_p \omega / \omega \ll 1 \). Assuming \( \omega = kv_0 + \omega(1) \), \( \omega_p = \omega_p(0) + \omega(1) \), we find

\*

The functions \( f_i \) have a spatial period which is the same as the period of the wave, and leads to a variation of the amplitude at characteristic times \( \Delta t \gg \omega_p^{-1} \), also assuming the condition \( \langle f_i \rangle = 0 \).

\[\text{We note that the approximation of spatially uniform wave fields is justified when the variation of the wave amplitudes and of the averages takes place sufficiently rapidly, and the derivatives of these quantities with respect to } x \text{ can be neglected.} \]