Sums of Generators of Analytic Semigroups
and Multivalued Linear Operators (*).

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Abstract. – Let A and B be generators of analytic semigroups in a Banach space. Under some conditions on the commutator of the resolvents of A and B, already considered in the literature and not implying relative boundedness, we prove that the closure of $A + B$ (or a proper extension of it) also generates an analytic semigroup, and we characterize interpolation spaces related to it. As a tool, we use approximation and interpolation results for multivalued linear operators.

1. – Introduction.

The main aim of this paper is to present some perturbation results for generators of analytic semigroups. As a technical tool, we need to prove some facts concerning multivalued linear operators, which have an interest in themselves and constitute another aim of this paper.

By generator of an analytic semigroup (sometimes called sectorial operator) we mean a linear operator $A$ in a complex Banach space $X$ with (not necessarily dense) domain $D_A$ such that its resolvent set contains a sector of the complex plane $\Sigma = \{ z \in \mathbb{C} : z \neq \omega, |\text{arg}(z - \omega)| < \pi - \theta \}$, for some $\omega \in \mathbb{R}$, $\theta \in (0, \pi/2)$, and its resolvent family satisfies the estimate $\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq K|\lambda - \omega|^{-1}$ for some $K > 0$ and for every $\lambda \in \Sigma$. Under these conditions one can define in a usual way a semigroup $e^{tA} \in \mathcal{L}(X)$, not necessarily strongly continuous at $t = 0$ (see for instance [14]). We are interested in the additive perturbation problem: assuming that $A$ is a generator and $B$ is another linear operator in $X$, one looks for conditions implying that $A + B$ (or some other operator intrinsically related to $A + B$) is still a generator. We refer the reader to [13] and its bibliography for an exposition of the classical results, and to [10] for a more general study of the subject. Here we want to present some new theorems about this problem. They differ substantially from the classical ones, because we never assume the usual relative boundedness condition $D_A \subset D_B$ which, often supplemented with some quantitative assumptions, states that the operator $B$ is in some sense

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«small» with respect to $A$. We will be able to allow «large» perturbations, since we are not trying to weaken this condition, but rather to replace it by entirely different (and independent) assumptions. Also, we do not assume dissipativeness nor self-adjointness for $A$ and $B$. We list our assumptions (in a rough form).

\begin{align*}
\text{(H1.S)} \quad & A \text{ and } B \text{ are linear operators in } X \text{ with domains } D_A, D_B \text{ and there exist } \vartheta \in (0, \pi/2), c > 0 \text{ such that} \\
& \left\| (A - z)^{-1} \right\|_{L(X)} + \left\| (B - z)^{-1} \right\|_{L(X)} \leq c |z|^{-1} \\
& \text{for every } z \in \mathbb{C} \text{ with } |\arg(z)| < \pi - \vartheta.
\end{align*}

\begin{align*}
\text{(H2.DPG.S)} \quad & \text{We have } (A - v)^{-1}D_B \subset D_B \text{ and there exist } c > 0, \alpha, \beta \text{ such that} \\
& -1 \leq \alpha < \beta \leq 1, \\
& \left\| [B; (A - v)^{-1}](B - z)^{-1} \right\|_{L(X)} \leq c \frac{1}{|v|^{1-\alpha}|z|^\beta}, \\
& \text{for every } v, z \in \mathbb{C} \text{ with } |\arg(v)|, |\arg(z)| < \pi - \vartheta,
\end{align*}

(by $[P; Q]$ we denote the commutator $PQ - QP$ of two linear operators in $X$).

Remark that (H1.S) simply states that both $A$ and $B$ generate bounded analytic semigroups. The main hypothesis (H2.DPG.S) is essentially taken from the paper [4] by G. Da Prato and P. Grisvard. They used such an assumption in order to find solutions (in various senses) of the equation

$$(A + B - \lambda)x = y$$

with $x$ (unknown) and $y$ (datum) in $X$, and $\lambda > 0$ sufficiently large. Their systematic treatment, together with many examples and applications they present, shows that (H2.DPG.S) is fairly natural. Another way to look at (H2.DPG.S) is the following. Consider the commutativity hypothesis

\begin{align*}
\text{(H2.COMM)} \quad & [(A - v)^{-1}; (B - z)^{-1}] = 0.
\end{align*}

If (H2.COMM) holds, the semigroups $e^{tA}$ and $e^{tB}$ commute, so that their product is still a semigroup. So in this case the simplest way to define a semigroup generated by $A + B$ is simply to identify it with $e^{tA}e^{tB}$ (a more precise characterization of the generator of such semigroup follows from our results below). So (H2.DPG.S) can be considered as the statement that the commutator of $A$ and $B$, though not zero, is suitably «small», so that our situation is a perturbation (and an extension) of the commutative case.

In the following, we give a short account of our results, sometimes omitting some details. First we can prove:

\begin{itemize}
\item[(i)] Assume (H1.S), (H2.DPG.S) and suppose that $D_A$ and $D_B$ are dense in $X$. Then $A + B$ is closable and its closure $\overline{A + B}$ generates a strongly continuous analytic semigroup.
\end{itemize}