ESTIMATES OF REMAINDER TERMS IN THE THEORY OF PERTURBED
DIRICHLET CHARACTERS

I. V. Elistratov

Lower and upper estimates for a summatory function of the principal Dirichlet character, perturbed on a finite set of prime numbers, are considered. It is shown that in the case of perturbation on a prime number, the remainder terms in these lower and upper estimates have one and the same order.

One of the directions of development of the theory of characters is the extension of the results of the theory of the Dirichlet characters (in particular, of the theory of Dirichlet L functions) to the characters, near to the Dirichlet characters in some sense. In this connection the notion of the perturbation of a Dirichlet character is introduced.

Definition 1. An arithmetic function $h(n)$ is called a character if it satisfies the following conditions:
1) $h(n)$ is not identically equal to zero.
2) $h(n)$ is a completely multiplicative function, i.e.,
   \[ h(n_1 \cdot n_2) = h(n_1) \cdot h(n_2) \]
   for arbitrary natural numbers $n_1$ and $n_2$.
3) $|h(n)| = 0$ or 1 (normalizedness).

Definition 2. Let $\chi(n)$ be a Dirichlet character. A character $h(n)$ is called a perturbed Dirichlet character of $\chi(n)$ if $h(n)$ is not a Dirichlet character and the set of prime numbers $p$, for which $h(p) \neq \chi(p)$, is such that
   \[ \sum_{p, h(p) \neq \chi(p)} 1/p < \infty. \]

Let $h(n)$ be the principal Dirichlet character $\chi_0(n, D)$, perturbed on a finite set of prime numbers. It is sufficient to consider the case where the perturbation takes place only on prime numbers which are not divisors of the modulus $D$. In the contrary case $h(n)$ can be considered as a perturbed character for the principal Dirichlet character $\chi_0(n, D_1)$, where $D_1/D$.

Theorem 1. Let a character $h(n)$ be obtained by the perturbation of the principal Dirichlet character $\chi_0(n, D)$ on the prime numbers $p_1, p_2, \ldots, p_m$ which are not divisors of the modulus $D$. Then
   \[ \sum_{n=1}^{\infty} h(n) = \Pi_p (1 - 1/p) (1 - h(p)/p)^{-1} \cdot x + O (\ln^m x). \]

Proof. Let us introduce the Dirichlet generating series of the function $h(n)$
   \[ L(s, h) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}. \]
If, as usual, $s = \sigma + it$, then for $\sigma > 1$ function $L(s, h)$ can be decomposed in the Euler product
   \[ L(s, h) = \Pi_p (1 - h(p)/p^s)^{-1}. \]
Hence we get
   \[ L(s, h) = L(s, \chi_0) \cdot F(s), \]
\[(1)\]
where $L(s, \chi_0)$ is the Dirichlet L-series with the principal character $\chi_0(n)$ and

$$F(s) = \prod_{i=1}^{m} (1 - 1/p_i^s) (1 - h(p_i)/p_i)^{-1}.$$  

It is clear that

$$F(s) = \prod_{i=1}^{m} \left(1 + (h(p_i) - 1)/p_i^s + h(p_i)(h(p_i) - 1)/p_i^{2s} + \ldots\right) = \sum_{n=1}^{\infty} b(n)/n^s$$

(the last equality in this chain is the definition of the multiplicative function $b(n)$).

Comparison of coefficients in Eq. (1) gives

$$h(n) = \sum_{d|n} b(d) \chi_0(n/d).$$

Hence it follows that

$$\sum_{h \leq x} h(n) = \sum_{h \leq x} \sum_{d \leq h} b(d) \chi_0(n/d) = \sum_{d \leq x} b(d) \sum_{h \leq x} \chi_0(l).$$

Since

$$\sum_{h \leq x} \chi_0(n) = (\varphi(D)/D) \cdot x + O(1),$$

it follows that

$$\sum_{h \leq x} h(n) = (\varphi(D)/D) \cdot x \sum_{d \leq x} b(d)/d + O(\sum_{d \leq x} |b(d)|).$$

We define the multiplicative function $b^*(n)$ by the equation

$$\prod_{i=1}^{m} (1 + 2p_i^{s^r} + 2p_i^{-2s^r} + \ldots) = \sum_{n=1}^{\infty} b^*(n)/n^s.$$  

It is obvious that the function $b^*(n) \geq 0$. Moreover, since

$$|h^i(p_i)(h(p_i) - 1)| \leq 2$$

for arbitrary $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots$, it follows that $|b(n)| \leq b^*(n)$ for every natural number $n$. Therefore, we can rewrite Eq. (2) in the following form:

$$\sum_{h \leq x} h(n) = (\varphi(D)/D) \cdot x \sum_{d \leq x} b(n)/d + O(\sum_{d \leq x} b^*(n)).$$

We will prove that

$$\sum_{h \leq x} b^*(n) = O(ln^n x).$$

(3)

For this we introduce the arithmetic functions $b^*_r(n)$, $r = 1, 2, \ldots, m$, defined by the equations

$$\prod_{i=1}^{m} (1 + 2p_i^{s^r} + 2p_i^{-2s^r} + \ldots) = \sum_{n=1}^{\infty} b^*_r(n)/n^s.$$  

Thus, $b^*(n) = b^*_m(n)$. We will prove by induction that

$$\sum_{n \leq x} b^*_r(n) = O(ln^r x)$$

(4)

for arbitrary $r = 1, 2, \ldots, m$. By the same token the estimate (3) will be established. For $r = 1$, it is obvious that

$$\sum_{n \leq x} b^*_1(n) = 1 + 2 \sum_{p_i^j \leq x} 1 = O(ln^x x).$$

Further, it follows from the definition of $b^*_r(n)$ that

$$b^*_r(n) = \sum_{d \leq x} \lambda_r(d) b^*_{r-1}(n/d),$$

where $L(s, \chi_0)$ is the Dirichlet L-series with the principal character $\chi_0(n)$ and

$$F(s) = \prod_{i=1}^{m} (1 - 1/p_i^s) (1 - h(p_i)/p_i)^{-1}.$$  

It is clear that

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(the last equality in this chain is the definition of the multiplicative function $b(n)$).

Comparison of coefficients in Eq. (1) gives

$$h(n) = \sum_{d|n} b(d) \chi_0(n/d).$$

Hence it follows that

$$\sum_{h \leq x} h(n) = \sum_{h \leq x} \sum_{d \leq h} b(d) \chi_0(n/d) = \sum_{d \leq x} b(d) \sum_{h \leq x} \chi_0(l).$$

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