Minimum Degree of Bipartite Graphs and the Existence of \(k\)-Factors*

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Abstract. Let \(G\) be a bipartite graph with bipartition \((X, Y)\) and \(k\) a positive integer. If

(i) \(|X| = |Y|\),

(ii) \(\delta(G) \geq \left\lceil \frac{|X|}{2} \right\rceil \geq k\),

(iii) \(|X| \leq 4k - 4\sqrt{k} + 1\) when \(|X|\) is odd and \(|X| \geq 4k - 2\) when \(|X|\) is even,

then \(G\) has a \(k\)-factor.

All graphs considered are assumed to be simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let \(G\) be a graph. For any set \(S\) of vertices of \(G\), we define the neighbour set of \(S\) in \(G\) to be the set of all vertices adjacent to vertices in \(S\); this set is denoted by \(N(S)\). If \(S\) and \(T\) are disjoint sets of vertices of \(G\), we write \(e(S, T)\) for the number of edges of \(G\) joining \(S\) to \(T\).

A vertex-function of \(G\) is a mapping \(f\) of the vertex-set \(V(G)\) into the set of positive integers. Given such a function \(f\), we define an \(f\)-factor of \(G\) a spanning subgraph \(F\) of \(G\) such that

\[d_F(x) = f(x)\]

for every vertex \(x\).

Now suppose that \(G\) is bipartite with bipartition \((X, Y)\). We say that \(f\) is balanced with respect to \((X, Y)\) if

\[\sum_{x \in X} f(x) = \sum_{y \in Y} f(y)\]

Ore and Ryser [3] proved the following theorem.

Ore-Ryser \(f\)-factor theorem. Let \(G\) be a bipartite graph with bipartition \((X, Y)\) and suppose that \(f\) is balanced with respect to \((X, Y)\). Then \(G\) has no \(f\)-factor if and only if there exists a subset \(T\) of \(Y\) such that

\[\sum_{t \in T} f(t) > \sum_{x \in X} \min(f(x), e(T, x)).\]

In this paper we only deal with \(k\)-factors i.e., \(f(x) = k\) for all vertices of \(G\). Thus Ore-Ryser theorem becomes:

* This work is dedicated to Astero.
Ore-Ryser $k$-factor theorem. Let $G$ be a bipartite graph with bipartition $(X, Y)$ such that $|X| = |Y|$. Then $G$ does not have a $k$-factor if and only if there exists a subset $T$ of $Y$ such that

$$k|T| > r_1 + 2r_2 + \cdots + k(r_k + \cdots + r_d)$$

where $R_i = \{x \in X | e(x, T) = i\}, r_i = |R_i|$ and $\Delta$ is the maximum degree of $G$.

The following theorem was obtained in [2] and examines the relation between the minimum degree of a graph and the existence of a $k$-factor.

**Theorem 1:** Let $G$ be a graph and $k$ a positive integer such that

(i) $k|V(G)|$ is even,
(ii) $\delta(G) \geq \left\lceil \frac{|V(G)|}{2} \right\rceil \geq k$,
(iii) $|V(G)| \geq 4k - 5$.

Then $G$ has a $k$-factor.

In this paper we prove the following similar result for bipartite graphs.

**Theorem 2:** Let $G$ be a bipartite graph with bipartition $(X, Y)$ and $k$ a positive integer.

If

(i) $|X| = |Y|$, 
(ii) $\delta(G) \geq \left\lceil \frac{|X|}{2} \right\rceil \geq k$,
(iii) $|X| \geq 4k - 4\sqrt{k} + 1$ when $|X|$ is odd and $|X| \geq 4k - 2$ when $|X|$ is even,

then $G$ has a $k$-factor.

**Lemma 3:** Let $G$ be a bipartite graph with bipartition $(X, Y)$, $k$ a positive integer and let $T$ be a subset of $Y$. Define $R_i = \{x \in X | e(x, T) = i\}, |R_i| = r_i$ for $0 \leq i \leq \Delta$, where $\Delta = \Delta(G)$, and let $m = \min\{i | R_i \neq \emptyset\}$.

If

(i) $|X| = |Y|$, 
(ii) $\delta(G) \geq \left\lceil \frac{|X|}{2} \right\rceil \geq k$,
(iii) $|X| \geq 4k - 4\sqrt{k} + 1$ when $|X|$ is odd and $|X| \geq 4k - 4\sqrt{k} + 2$ when $|X|$ is even,
(iv) $k|T| > \sum_{i=1}^{\Delta} i r_i + k \sum_{i=k}^{\Delta} r_i$,

then

(a) $|T| \leq |X| - \delta(G) + m$,
(b) $N(T) = X$ and $m \leq k - 1$,
(c) $\sum_{i=k}^{\Delta} r_i \geq \left\lceil \frac{|X|}{2} \right\rceil + m$ and $\sum_{i=k}^{\Delta} r_i \leq \left\lceil \frac{|X|}{2} \right\rceil - m$.

**Proof.** (a) Let $u$ be an element of $R_m$. Then since $|N(u) \cap T| = m, |N(u) \cap (Y \setminus T)| \geq \delta(G) - m$. Thus $|Y| - |T| \geq \delta(G) - m$. So $|T| \leq |X| - \delta(G) + m$.

(b) Suppose that $N(T) \neq X$. Thus $m = 0$ and so by (a), $|T| \leq |X| - \delta(G)$. But since by hypothesis $\delta(G) \geq \left\lceil \frac{|X|}{2} \right\rceil$, we have that $|T| \leq \left\lceil \frac{|X|}{2} \right\rceil$ and hence,