Decomposition of the Complete $r$-Graph into Complete $r$-Partite $r$-Graphs*

Noga Alon

Department of Mathematics, Tel Aviv University, Tel Aviv, Israel, and Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Abstract. For $n \geq r \geq 1$, let $f_r(n)$ denote the minimum number $q$, such that it is possible to partition all edges of the complete $r$-graph on $n$ vertices into $q$ complete $r$-partite $r$-graphs. Graham and Pollak showed that $f_2(n) = n - 1$. Here we observe that $f_3(n) = n - 2$ and show that for every fixed $r \geq 2$, there are positive constants $c_1(r)$ and $c_2(r)$ such that $c_1(r) \leq f_r(n) \leq c_2(r)$ for all $n \geq r$. This solves a problem of Aharoni and Linial. The proof uses some simple ideas of linear algebra.

1. Introduction

For $n \geq r \geq 1$, let $f_r(n)$ denote the minimum number $q$, such that it is possible to partition all edges of the complete $r$-uniform hypergraph on $n$ vertices into $q$ pairwise edge-disjoint complete $r$-partite $r$-uniform hypergraphs.

Obviously, $f_1(n) = 1$. Graham and Pollak ([3, 4], see also [2, 5]) proved that $f_2(n) = n - 1$ for all $n \geq 2$. Simple proofs for this result were found by Tverberg [7] and Peck [6].

Aharoni and Linial [1] raised the natural problem of determining or estimating $f_r(n)$ for $r > 2$. In particular they asked if $f_r(n)$ is a nonlinear function of $n$, for some fixed $r > 2$.

In this note we answer this question in the affirmative by proving the following theorem, that determines the asymptotic behavior of $f_r(n)$ for every fixed $r$ as $n$ tends to infinity.

Theorem 1.1. For every fixed $r \geq 1$, there are two positive constants $c_1 = c_1(r)$ and $c_2 = c_2(r)$ such that

$$c_1 \cdot n^{\lfloor r/2 \rfloor} \leq f_r(n) \leq c_2 \cdot n^{\lfloor r/2 \rfloor}$$

for all $n \geq r$.

The lower bound is proved using some simple ideas of linear algebra. The method is similar to the one used by Tverberg [7] and by Graham and Pollak [3, 4], for determining $f_2(n)$. The upper bound is established by a recursive construction.

* Research supported in part by Air Force Contract OSR 82-0326 and by Alon Fellowship.
It is worth noting that our construction supplies the exact value of $f_3(n) = n - 2$ for all $n \geq 3$.

2. The Lower Bound

We start with the following easy observation.

**Lemma 2.1.** For every $n \geq r \geq 2$

$$f_r(n) \geq f_{r-1}(n-1).$$

**Proof.** Suppose all edges of the complete $r$-uniform hypergraph on a set $N = \{1, 2, \ldots, n\}$ of $n$ vertices are partitioned into $q = f_2(n)$ $r$-partite $r$-graphs (= $r$-uniform hypergraphs) $H^1, H^2, \ldots, H^q$. Let $E_i$ denote the set of edges of $H^i$ and put $E_i = \{e \setminus \{n\} : e \in E_i, n \in e\}$. Clearly each nonempty $E_i$ is the set of edges of a complete $(r-1)$-partite $(r-1)$-graph. Moreover, the set of all nonempty $E_i$’s forms a decomposition of all edges of the complete $r-1$-uniform hypergraph on the $n-1$ vertices $N \setminus \{n\}$. Hence $f_{r-1}(n-1) \leq q = f_2(n)$, as needed. □

In view of Lemma 2.1, the lower bound in Theorem 1.1 for odd values of $r$ follows from the lower bound for even values of $r$, which we prove next.

**Lemma 2.2.** For all $n \geq 2k \geq 2$

$$f_{2k}(n) \geq 2 \cdot \binom{n}{2k} - \binom{n}{k} + \binom{n}{k-1} - \binom{n}{k-3} - \cdots - \binom{n}{k + 1 - 2 \cdot \lfloor k/2 \rfloor}.$$ 

**Proof.** Let $K = \{K \subset N : |K| = k\}$ be the set of all $\binom{n}{k}$-subsets of $N = \{1, 2, \ldots, n\}$ and associate each $K \in K$ with a variable $x_{K}$. Let $H$ be a complete $2k$-partite $2k$-graph, whose (pairwise disjoint) vertex classes $V_1, V_2, \ldots, V_{2k}$ are subsets of $N$. By definition, the edges of $H$ are all $2k$-subsets $A \subset N$ such that $|A \cap V_i| = 1$ for $1 \leq i \leq 2k$. We define, for each such $H$, a quadratic form $Q(H)$ in the variables $\{x_K : K \in K\}$ as follows.

$$Q(H) = \sum \{L_a(H) \cdot L_b(H) : A, B \subset \{1, 2, \ldots, 2k\}, |A| = |B| = k, A \cap B = \emptyset, 1 \in A\},$$

where, for $C \subset \{1, 2, \ldots, 2k\}$, $|C| = k$,

$$L_C(H) = \sum \{x_K : K \in K, |K \cap V_c| = 1 \text{ for all } c \in C\}.$$ 

Thus, $Q(H)$ is a sum of $\binom{2k}{k}^2$ products of the form $L_a(H) \cdot L_b(H)$, in which each factor is a linear combination of the $x_K$’s.

Put $q = f_{2k}(n)$, and suppose the edges of the complete $r$-graph on $N$ are partitioned into $q$ $r$-partite $r$-graphs $H^1, H^2, \ldots, H^q$. One can easily check that

$$\sum_{i=1}^{q} Q(H^i) = \sum \{x_K \cdot x_L : K, L \in K, K \cap L = \emptyset\}. \quad (2.1)$$

Indeed, if $K, L \in K$ and $K \cap L = \emptyset$ then the product $x_K \cdot x_L$ appears only in $Q(H^i)$,