Paths and Edge-Connectivity in Graphs III.  
Six-Terminal $k$ Paths

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Abstract. Suppose that $k \geq 1$ is an odd integer, $(s_1, t_1), \ldots, (s_k, t_k)$ are pairs of vertices of a graph $G$ and $\lambda(s_i, t_i)$ is the maximal number of edge-disjoint paths between $s_i$ and $t_i$. We prove that if $\lambda(s_i, t_i) \geq k \ (1 \leq i \leq k)$ and $|\{s_1, \ldots, s_k, t_1, \ldots, t_k\}| \leq 6$, then there exist edge-disjoint paths $P_1, \ldots, P_k$ such that $P_i$ has ends $s_i$ and $t_i \ (1 \leq i \leq k)$.

1. Introduction

We consider finite undirected graphs possibly with multiple edges but without loops. Let $G$ be a graph and let $V = V(G)$ and $E = E(G)$ be the set of vertices and edges of $G$ respectively. For vertices $x$ and $y$, $\lambda(x, y) = \lambda(x, y; G)$ denotes the maximal number of edge-disjoint paths between $x$ and $y$, and a path $P = P[x, y]$ denotes a path between $x$ and $y$. Let $(s_1, t_1), \ldots, (s_k, t_k)$ be pairs of vertices of $G$, and let $T = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$. When is the following statement true?

$$\text{(1.1) There exist edge-disjoint paths } P_1[s_1, t_1], P_2[s_2, t_2], \ldots, P_k[s_k, t_k].$$


For integers $k \geq 1$ and $n \geq 2$, set

$$\lambda(k, n) := \min\{m : |T| \leq n \text{ and } \lambda(s_i, t_i) \geq m \ (1 \leq i \leq k), \text{ then (1.1) holds}\},$$

$$\lambda(k) := \lambda(k, 2k) = \lambda(k, m)(m > 2k).$$

The following results are known.

(1.2) $\lambda(1) = 1, \lambda(2) = 3, \lambda(k) \leq k + 2 \ (k = 3, 4, 5)$ (Cypher [1]).

(1.3) $\lambda(3) = 3$ (Okamura [4]).

(1.4) $\lambda(4) = 5, \lambda(k) \leq 2k - 3 \ (k \geq 5)$ (Hirata Kubota and Saito [2]).

(1.5) $\lambda(k, 3) = k, \lambda(k, 4) = \lambda(k, 5) = \begin{cases} k & \text{if } k \text{ is odd}, \\ k + 1 & \text{if } k \text{ is even} \end{cases}$ (Okamura [5]).

(1.6) If $k \geq 5$ is odd, $\lambda(s_i, t_i) \geq k \ (1 \leq i \leq k)$, $s_3 = s_4 = \cdots = s_k$ and $t_3 = t_4 = \cdots = t_k$, then (1.1) holds (Hirata, Kubota and Saito [2]).
Our main result is

**Theorem 1.** For each integer $k \geq 1$, 

$$\lambda(k, 6) = \begin{cases} k & \text{if } k \text{ is odd,} \\ k + 1 & \text{if } k \text{ is even.} \end{cases}$$

We consider the following statements in which $e(s_i) = e(s_i; G)$ denotes the degree of $s_i$.

(1.7) $k \geq 3$ is an odd integer and $\lambda(s_i, t_i) \geq k$ ($1 \leq i \leq k$).

(1.8) $k \geq 3$ is an odd integer, $e(s_i) = e(t_i) = \lambda(s_i, t_i) = k$ ($1 \leq i \leq k$), for each $x \in V - T$ $e(x) = 3$, and for each $f \in E \setminus (s_i, t_i; G - f) < k$ for some $i$.

(1.9) For some $1 \leq p < q \leq k$, there exist edge-disjoint paths $P_1[s_p, t_p]$ and $P_2[s_q, t_q]$ such that for each $1 \leq i \leq k$ with $i \neq p, q$, $\lambda(s_i, t_i; G - \cup_{j=1}^{q} E(P_j)) \geq k - 2$.

Using Mader [3] it is shown that if (1.8) implies (1.1), then (1.7) implies (1.1) (see the proof of Theorem 5 in [5]). When $|T| = 5$ (1.7) implies (1.9) ([5]), and so (1.1), but when $|T| = 6$, we have

**Theorem 2.** Suppose that

(i) $G$ is a graph, $T = \{s_1, \ldots, s_k, t_1, \ldots, t_k\} \subseteq V$ and $|T| \leq 6$.

(ii) (1.8) is true in $G$.

(iii) (1.9) is not true in $G$.

Then $|T| = 6$, for some integer $p \geq 3$, $k = 3p$ and $G$ is the graph in Fig. 1.1 with $\min(i, j - i, k - j) \geq 2$ (see also Fig. 1.2).

It easily follows that (1.1) is true in the graph in Fig. 1.1. Thus Theorem 2 implies Theorem 1 (see Theorem 3 in [5] for the case when $k$ is even).

**Notations and Definitions**

In all notations we often omit $G$. Let $k \geq 1$ be an integer, $X, Y \subseteq V, \{x, x_1, x_2, x_3, x_4, x_5\} \subseteq V$ and $f \in E$. We often denote $\{x\}$ by $x$ and $\{f\}$ by $f$. We set

![Fig. 1.1](image-url)