Characterization of \( \{v_{\mu+1} + 2v_{\mu}, v_{\mu} + 2v_{\mu-1}; t, q\} \)-min. hypers and Its Applications to Error-Correcting Codes

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Abstract. Recently, Hamada [5] characterized all \( \{v_2 + 2v_1, v_1 + 2v_0; t, q\} \)-min. hypers for any integer \( t \geq 2 \) and any prime power \( q \geq 3 \) where \( v_t = (q^t - 1)/(q - 1) \) for any integer \( t \geq 0 \). The purpose of this paper is to characterize all \( \{v_{\mu+1} + 2v_{\mu}, v_{\mu} + 2v_{\mu-1}; t, q\} \)-min. hypers for any integers \( t, \mu \) and any prime power \( q \) such that \( t \geq 3, 2 \leq \mu \leq t - 1 \) and \( q \geq 5 \) and to characterize all \( (n, k, d; q) \)-codes meeting the Griesmer bound (1.1) for the case \( k \geq 3, d = q^{k-1} - (2q^{\mu-1} + q^\mu) \) and \( q \geq 5 \) using the results in Hamada [3, 4, 5].

1. Introduction

Let \( F \) be a set of \( f \) points in a finite projective geometry \( PG(t, q) \) of \( t \) dimensions (cf. Appendix I) where \( t \geq 2, f \geq 1 \) and \( q \) is a prime power. If (a) \( |F \cap H| \geq m \) for any hyperplane \( H \) in \( PG(t, q) \) and (b) \( |F \cap H| = m \) for some hyperplane \( H \) in \( PG(t, q) \), then \( F \) is said to be an \( \{f, m; t, q\} \)-min. hyper (or an \( \{f, m; t, q\} \)-minihyper) where \( m > 0 \) and \( |A| \) denotes the number of elements in the set \( A \). In [5], the author characterized all \( \{v_2 + 2v_1, v_1 + 2v_0; t, q\} \)-min. hypers for any integer \( t \geq 2 \) and any prime power \( q \geq 3 \).

The purpose of this paper is to generalize the above result, i.e., to characterize all \( \{v_{\mu+1} + 2v_{\mu}, v_{\mu} + 2v_{\mu-1}; t, q\} \)-min. hypers for any integers \( t, \mu \) and any prime power \( q \) such that \( t \geq 3, 2 \leq \mu \leq t - 1 \) and \( q \geq 5 \) using the results in Hamada [3, 4, 5] where \( v_t = (q^t - 1)/(q - 1) \) for any integer \( t \geq 0 \) (cf. Theorems 2.2 and 2.3 in Section 2). Using those results, all \( (n, k, d; q) \)-codes (i.e., all \( q \)-ary linear codes with length \( n \), dimension \( k \), and minimum distance \( d \)) meeting the Griesmer bound:

\[
n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor
\]

(1.1)

will be characterized for the case \( k \geq 3, d = q^{k-1} - (2q^{\mu-1} + q^\mu) \) and \( q \geq 5 \) (cf. Theorem 2.4) where \( 1 \leq \mu \leq k - 2 \) and \( \lfloor x \rfloor \) denotes the smallest integer \( \geq x \).

2. Main Results

Let \( q \) be any prime power and let \( k \) and \( d \) be any integers such that \( k \geq 3 \) and \( 1 \leq d \leq q^{k-1} - q \). Then \( d \) can be expressed uniquely as follows:
\[ d = q^{k-1} - \left( e + \sum_{i=1}^{h} q^{\mu_i} \right) \]

using some ordered set \((e, \mu_1, \mu_2, \ldots, \mu_h)\) in \(U(k-1, q)\) where \(U(t, q)\) denotes the set of all ordered sets \((e, \mu_1, \mu_2, \ldots, \mu_h)\) of integers \(e, h\) and \(\mu_i\)'s such that \(0 \leq e \leq q - 1,\) \(1 \leq h \leq (t-1)(q-1),\) \(1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_h \leq t - 1\) and \(0 \leq n_l(\mu) \leq q - 1\) for \(l = 1, 2, \ldots, t - 1.\) Here \(n_l(\mu)\) denotes the number of integers \(\mu_i\) in \(\mu \equiv (\mu_1, \mu_2, \ldots, \mu_h)\) such that \(\mu_i = l\) for the given integer \(l.\) In this case, the bound (1.1) can be expressed as follows:

\[ n \geq v_k - \left( e + \sum_{i=1}^{h} v_{\mu_i+1} \right) \]

where \(v_k = (q^k - 1)/(q - 1)\) for any integer \(l \geq 0.\)

Let \(F_0(e, \mu_1, \mu_2, \ldots, \mu_h; t, q)\) be the family of all unions \(\bigcup_{i=0}^{h} V_i\) of a set \(V_0 \equiv \{P_1, P_2, \ldots, P_t\},\) a \(\mu_1\)-flat \(V_1,\) a \(\mu_2\)-flat \(V_2,\ldots,\) a \(\mu_h\)-flat \(V_h\) in \(PG(t, q)\) which are mutually disjoint where \(P_1, P_2, \ldots, P_t\) denote \(e\) points in \(PG(t, q)\) and \(V_0 = \emptyset\) in the case \(e = 0.\) As occasion demands, \(F_0(e, \mu_1, \mu_2, \ldots, \mu_h; t, q)\) will be denoted by \(\mathcal{F}(\lambda_1, \lambda_2, \ldots, \lambda_h; t, q)\) where \(\eta = h + e, \lambda_i = 0 (i = 1, 2, \ldots, e)\) and \(\lambda_{i+j} = \mu_j (j = 1, 2, \ldots, h).\)

**Remark 2.1.** It is known (cf. Theorems 2.2 and 2.3 in Hamada and Tamari [10] for example) that \(F_0(e, \mu_1, \mu_2, \ldots, \mu_h; t, q) \neq \emptyset\) if and only if either (I) \(h = 1\) or (II) \(h \geq 2\) and \(\mu_{h-1} + \mu_h = t - 1\) where \((e, \mu_1, \mu_2, \ldots, \mu_h) \in U(t, q).\)

Recently, the following theorem has been established by Hamada [5].

**Theorem 2.1.** (1) In the case \(t \geq 2\) and \(q \geq 5, F\) is a \(\{v_1 + 2v_2, v_1 + 2v_0; t, q\}\)-min. hyper if and only if \(F \in \mathcal{F}(0, 0, 1; t, q)\) where \(v_0 = 0, v_1 = 1\) and \(v_2 = q + 1.\)

(2) In the case \(t \geq 2\) and \(q = 3, F\) is a \(\{v_1 + 2v_2, v_1 + 2v_0; t, 3\}\)-min. hyper if and only if either (a) \(F \in \mathcal{F}(0, 0, 1; t, 3)\) or (b) \(F = \{(\omega_0), (\omega_0 + \omega_1), (2\omega_0 + \omega_1), (\omega_2), (\omega_0 + \omega_2), (\omega_0 + 2\omega_1 + \omega_2)\}\) for some noncollinear points \(\omega_0, \omega_1, \omega_2\) in \(PG(t, 3)\) and some integer \(c\) in \([1, 2]\) where \(v_0 = 0, v_1 = 1\) and \(v_2 = 4.\)

(3) In the case \(t \geq 2\) and \(q = 4, F\) is a \(\{v_1 + 2v_2, v_1 + 2v_0; t, 4\}\)-min. hyper if and only if either (a) \(F \in \mathcal{F}(0, 0, 1; t, 4)\) or (b) \(F = \{(\omega_0 + \omega_1), (\omega_0 + \omega_2), (\omega_0 + \omega_1 + \omega_2), (\omega_0 + 2\omega_1 + \omega_2), (\omega_0 + 2\omega_1 + \omega_2)\}\) for some noncollinear points \(\omega_0, \omega_1, \omega_2\) in \(PG(t, 4)\) and some element \(c\) in \([1, x, x^2]\) where \(v_0 = 0, v_1 = 1, v_2 = 5\) and \(x\) is a primitive element of \(GF(2^2)\) such that \(x^2 = x + 1\) and \(x^3 = 1.\)

The purpose of this paper is to generalize Theorem 2.1, i.e., to prove the following two theorems whose proofs will be given in Sections 3 and 4.

**Theorem 2.2.** Let \(\mu\) and \(q\) be any integer \(\geq 2\) and any prime power \(\geq 5\), respectively. (1) In the case \(t \geq 2\mu, F\) is a \(\{v_{\mu+1} + 2v_{\mu}, v_\mu + 2v_{\mu-1}; t, q\}\)-min. hyper if and only if \(F \in \mathcal{F}(\mu - 1, \mu - 1, \mu; t, q).\)

(2) In the case \(\mu < t < 2\mu,\) there is no \(\{v_{\mu+1} + 2v_{\mu}, v_\mu + 2v_{\mu-1}; t, q\}\)-min. hyper.

**Theorem 2.3.** Let \(t\) and \(\mu\) be any integers such that \(t > \mu \geq 3\) and let \(q = 3\) or \(4.\) If \(F^* \in \mathcal{F}(1, 1, 2; t, q)\) for any \(\{v_3 + 2v_2, v_2 + 2v_1; t, q\}\)-min. hyper \(F^*,\) then