For a large class of plane domains $\Omega$, having exits at infinity, one shows the coincidence of the spaces of solenoidal vector fields $\bar{f}_1^\prime(\Omega)$ and $\bar{f}_2^\prime(\Omega)$, which play an important role in the investigation of initial-boundary-value problems for the Navier-Stokes equations.

Let $\Omega$ be a domain in $\mathbb{R}^2$. By $\mathcal{C}^\infty(\Omega)$ we shall denote the set of infinitely differentiable vector fields $\overline{U}(x) = (u_1(x), u_2(x))$ with compact supports, lying in $\Omega$, while by $\mathcal{F}^\infty(\Omega)$ we denote the set of all those $\overline{U} \in \mathcal{C}^\infty(\Omega)$, such that $\text{div} \overline{U} = 0$. Further, $\mathcal{W}_2^1(\Omega)$ is the closure of the set $\mathcal{C}^\infty(\Omega)$ in the norm

$$||\overline{U}|| = \left( \int_\Omega \left( |\overline{U}|^2 + |\nabla \overline{U}|^2 \right) dx \right)^{1/2}$$

(1)

$\bar{f}_1^\prime(\Omega)$ is the closure of the set $\mathcal{F}^\infty(\Omega)$ in the norm (1);

$\mathcal{F}_2^\prime(\Omega)$ is the subspace of the space $\mathcal{W}_2^1(\Omega)$ consisting of all those $\overline{U} \in \mathcal{W}_2^1(\Omega)$ for which $\text{div} \overline{U} = 0$. The spaces $\mathcal{W}_2^1(\Omega)$, $\bar{f}_1^\prime(\Omega)$ and $\mathcal{F}_2^\prime(\Omega)$ are Hilbert spaces with the inner product

$$\langle \overline{U}, \overline{V} \rangle = \int_\Omega (\overline{U} \cdot \nabla \overline{V} - \overline{V} \cdot \nabla \overline{U}) dx$$

where $\overline{U} \cdot \nabla = \sum_{i=1}^2 u_i \frac{\partial}{\partial x_i}$.

Obvious inclusions: $\bar{f}_1^\prime(\Omega) \subset \mathcal{F}_2^\prime(\Omega) \subset \mathcal{W}_2^1(\Omega)$.

In a series of questions related with the correct formulation and the solvability of boundary and initial-boundary-value problems for the Navier-Stokes equations, it is important to know whether the spaces $\bar{f}_1^\prime(\Omega)$ and $\mathcal{F}_2^\prime(\Omega)$ coincide or not (see [1, Sec. 1]). In [1] one proves that the spaces $\bar{f}_1^\prime(\Omega)$ and $\mathcal{F}_2^\prime(\Omega)$ coincide for a large class of domains with compact boundaries. We shall be interested in the case of plane domains $\Omega$ with noncompact boundaries, having exits at infinity, i.e., representable in the form of the union of a bounded domain $\Omega_0$ and pairwise disjoint unbounded domains $\Omega_1, ..., \Omega_N$, the exits at infinity. It has been proved in [1, 2] that $\bar{f}_1^\prime(\Omega) = \mathcal{F}_2^\prime(\Omega)$ if each exit at infinity of the domain $\Omega$ contains some nonzero angle [1, Theorem 4.2] or it is symmetric relative to some ray and has a global Lipschitz boundary [2, Theorem 7]. In [1, 2], the proof of the equality $\bar{f}_1^\prime(\Omega) = \mathcal{F}_2^\prime(\Omega)$ is carried out according to the following scheme. Since $\bar{f}_2^\prime(\Omega) \subset \mathcal{F}_2^\prime(\Omega)$, it is sufficient to show that $\bar{f}_1^\prime(\Omega) \subset \mathcal{F}_2^\prime(\Omega)$ and this follows from the following two propositions whose validity is established in [1, 2] for the classes of domains considered in these papers.

**PROPOSITION 1.** Let \( \mathbf{U} \in \mathcal{J}_1^p(\Omega) \). Then the flow of the field \( \mathbf{U} \) across any cross section of the domain \( \Omega \) is equal to zero.

**PROPOSITION 2.** Assume that \( \mathbf{U} \in \mathcal{J}_1^p(\Omega) \) and the flow across any cross section of the domain \( \Omega \) is equal to zero. Then \( \mathbf{U} \in \mathcal{J}_1^p(\Omega) \).

In the present paper we prove Propositions 1 and 2 and thus also the equality of the spaces \( \mathcal{J}_1^p(\Omega) \) and \( \mathcal{J}_2^p(\Omega) \) for a considerably larger class of domains \( \Omega \) than in [1, 2].

Thus, assume that the domain \( \Omega \subset \mathbb{R}^n \) can be represented in the form of a union of a bounded domain \( \Omega_0 \) and of pairwise disjoint exits at infinity \( \Omega_0, ..., \Omega_N \).

**THEOREM 1.** We assume that the boundary of each domain \( \Omega_i, 0=1, ..., N \), has one and only one noncompact connected component \( \Gamma_i \) and that \( \Gamma_i \) is a Jordan curve dividing the entire plane into two parts. Then, Proposition 1 holds for the domain \( \Omega \).

**Remark.** The boundary \( \partial \Omega \) of the domain \( \Omega \) may have an arbitrary number of compact components, bounded by "holes." Here we do not make any assumption regarding the smoothness of the compact components of the boundary.

**Proof of the Theorem.** Let \( \mathbf{U} \in \mathcal{J}_1^p(\Omega) \) and let \( \ell \) be an arbitrary cross section of the domain \( \Omega \), i.e., \( \ell \) is a smooth curve, lying entirely in \( \Omega \), with the exception of the endpoints \( P' \) and \( P'' \), belonging to different noncompact components of the boundary \( \partial \Omega \). We show that the flow of the field \( \mathbf{U} \) across the cross section \( \ell \) is equal to zero, i.e.,

\[
\int_{\ell} \mathbf{U} \cdot \mathbf{n} \, dl = 0
\]

where \( \mathbf{n} \) is the unit vector of the normal to the curve \( \ell \). Since \( \text{div} \mathbf{U} = 0 \), for the proof of (2) one can assume that the cross section \( \ell \) lies in some exit to infinity, say in \( \Omega_i \), and its endpoints \( P' \) and \( P'' \) belong to different noncompact components of \( \partial \Omega \cap \partial \Omega_i \).

We shall assume that the field \( \mathbf{U} \) is extended by zero in \( \mathbb{R}^n \setminus \Omega \). Then, obviously, \( \mathbf{U} \in \mathcal{J}_1^p(\mathbb{R}^n) \). Together with the field \( \mathbf{U} \), it will be convenient to consider the corresponding stream function \( \Psi \), defined on all of \( \mathbb{R}^n \) by the following formula:

\[
\Psi(x) = \int_0^1 (x_2 u_1(tx) - x_1 u_2(tx)) \, dt + \text{const.}
\]

In this case

\[
u_1(x) = \frac{\partial \Psi}{\partial x_2}, \quad u_2(x) = -\frac{\partial \Psi}{\partial x_1}
\]

since \( \text{div} \mathbf{U} = 0 \).

From (3), (4) and from the fact that the field \( \mathbf{U} \) belongs to the space \( \mathcal{J}_1^p(\mathbb{R}^n) \) it follows that the stream function \( \Psi \) is locally square summable in \( \mathbb{R}^n \), has generalized first and second-order derivatives which are square summable in all of \( \mathbb{R}^n \), and then, according to S. L. Sobolev's embedding theorem, the function \( \Psi \) is continuous in \( \mathbb{R}^n \). In addition, \( \Psi \)