Suppose a finite group $G$ is the product of a subgroups $A$ and $B$ of coprime orders, and suppose the order of $A$ is $p^aq^b$, where $p$ and $q$ are primes, and $B$ is a 2-decomposable group of even order. Assume that a Sylow $p$-subgroup $P$ is cyclic. If the order of $P$ is not equal to 3 or 7, then $G$ is solvable. If $G$ is nonsolvable and $G$ contains no nonidentity solvable invariant subgroups, then $G$ is isomorphic to $PSL(2, 7)$ or $PGL(2, 7)$.

In [1] the author described the finite nonsolvable groups which are a product of two subgroups of coprime orders, one of which is a Schmidt group and the other a 2-decomposable group (see also [2, pp. 70-100]). All properties of a Schmidt group are well known, in particular, it is biprimary, i.e., its order is divisible by exactly two distinct primes, and it has a nonidentity cyclic Sylow subgroup.

Developing the above-mentioned result of [1], we will prove in the present note the following.

**THEOREM 1.** Suppose a finite group $G$ is the product of subgroups $A$ and $B$ of coprime orders, and suppose that $A$ is a biprimary group and that $B$ is a 2-decomposable group of even order. Assume that $A$ contains a nonidentity cyclic Sylow subgroup $P$. If $G$ is nonsolvable, then $G/R(G)$ is isomorphic to $PSL(2, 7)$ or $PGL(2, 7)$.

Here $R(G)$ denotes the product of all solvable invariant subgroups of $G$.

**COROLLARY.** Suppose a group $G$ possesses the factorization mentioned in Theorem 1. If the order of $P$ is not equal to 3 or 7, then $G$ is solvable.

The proof of Theorem 1 begins with the study of the special case where $B$ is primary. This case, without the assumption that the order of $B$ is even, is described in

**THEOREM 2.** Suppose a nonsolvable group $G$ is the product of a biprimary subgroup $A$ and a primary subgroup $B$. If $G$ contains a cyclic Sylow subgroup, then $G/R(G)$ is isomorphic to one of the following:

1) $PSL(2, 5) = A_5; Z_5$;
2) $PSL(2, 7) = S_4Z_7 = (Z_7 \times Z_3)D_6$;
3) $PSL(2, 8) = (E_2 \times Z_2)Z_6$;
4) $PGL(2, 5) = S_4Z_5$;
5) $PGL(2, 7) = (Z_7 \times Z_3)D_6$;
6) $PGL(2, 8) = (E_2 \times Z_2)Z_3$ where $G_3$ is a Sylow 3-subgroup;
7) $PSL(3, 3)$, the order of $A$ is $2^43^3$, and $B \cong Z_{11}$.

Since biprimary groups are solvable, the group $G$ in Theorem 2 has an order divisible by exactly three distinct primes. Such simple groups are known at present only in the case where they contain a cyclic Sylow subgroup. This invites the requirement of cyclicity of a Sylow subgroup in the hypothesis of Theorem 1, hence also in the hypothesis of Theorem 1.

If we knew all simple groups of order $p^aq^br^c$, where $p$, $q$, and $r$ are distinct primes, then the method of proof of Theorem 1 would enable us to describe the nonsolvable groups with the factorization indicated in Theorem 1, without assuming cyclicity of $P$. 

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We will use the following notation: $S_n$ and $A_n$ are the symmetric and alternating groups of degree $n$; $Z_n$, $E_n$, and $D_n$ are the cyclic, elementary Abelian, and dihedral groups of order $n$. A semidirect product of groups $X$ and $Y$ with invariant subgroup $X$ will be denoted by $X \rtimes Y$. A group is called primary if its order is a power of a prime.

1. Preliminary Lemmas

**Lemma 1.** If a group $G$ is the product of two subgroups $A$ and $B$ of coprime orders and $K$ is a subinvariant subgroup of $G$, then $K = (K \cap A)(K \cap B)$.

**Proof.** If $N$ is an invariant subgroup of $G$, then $N \cap A$ is a Hall $\pi$-subgroup of $N$, where $\pi = \pi(A)$, and $N \cap B$ is a Hall $\pi'$-subgroup of $N$ (see [3, p. 35]). Therefore, $N = (N \cap A)(N \cap B)$. If $M$ is an invariant subgroup of $N$, then we again have

$$M = (M \cap N \cap A)(M \cap N \cap B) = (M \cap A)(M \cap B)$$

and so on.

**Lemma 2.** If a group $G$ is the product of a primary subgroup of odd order and a 2-decomposable subgroup, then $G$ is solvable.

**Proof.** Suppose $G = AB$, $A$ is a $p$-group, $p$ is an odd prime, and $B$ is a 2-decomposable group. The group $G$ contains a Sylow $p$-subgroup $P$ such that $P = AS$, where $S$ is some Sylow $p$-subgroup of $B$ (see [3, p. 676]). Since $B$ is solvable, we have $B = B_1S$, where $B_1$ is a Hall $p'$-subgroup of $B$. But now $G = AB = PB = PB_1$. By Burnside's lemma (see [3, p. 491]), $G$ is not simple. By Lemma 1, an invariant subgroup $N$ of $G$ is factorable, i.e., $N = (P \cap N)(B_1 \cap N)$, hence $N$ is solvable by induction. The factor group $G/N$ is also solvable by induction. Therefore, $G$ is solvable.

**Lemma 3.** The groups $PSU(3, 3^s)$ and $PSp(4, 3)$ do not contain biprimary Hall subgroups.

**Proof.** Suppose $G = PSU(3, 3^s)$. Then the order of $G$ is $2^{5s^2}3^s$ and a Sylow $7$-subgroup of $G$ is self-centralizing. Since the order of $G$ is greater than that of $S_7$, it follows that $G$ contains no subgroup of order $2^{5s^2}$.

Assume there exists a subgroup $K$ of order $3^{2s}$ by Sylow's theorem on the number of Sylow subgroups, $K$ is $7$-closed, i.e., a subgroup $P$ of $K$ of order $7$ is invariant in $K$. But now $K/P$ is isomorphic to a subgroup of the group of all automorphisms of $P$, which is isomorphic to $Z_{2s}$. Contradiction.

Assume there is a subgroup $H$ of order $2^{5s}$. As in the previous case, $H$ cannot be $7$-closed. Since the index in $H$ of the normalizer $N$ of a Sylow $7$-subgroup is congruent to $1$ modulo $7$, it follows that $|H:N| = 2^{5s}$ and $|N| = 2^{5s}$. Therefore, the order of $Z_{2s}$ must be divisible by $4$, which is impossible. Thus, $PSU(3, 3^s)$ contains no biprimary Hall subgroups.

Now suppose $G = PSp(4, 3)$. Then the order of $G$ is $2^{5s^2}3^s$, a Sylow $3$-subgroup $T$ of $G$ is non-Abelian, and $|T'| \leq 3^s$. A Sylow $2$-subgroup $R$ is also non-Abelian and $R/R'$ has exponent $2$. The normalizer of a Sylow $5$-subgroup $P$ of $G$ has order $20$, and the centralizer of $P$ in $G$ coincides with $P$ (see [3, pp. 229–231]).

Assume there exists a subgroup $H$ of order $3^{2s}$. Then $H$ is $3$-closed and, since $H$ is non-nilpotent, $N_{H}(P) = P$. The subgroup $T$ is non-Abelian, hence a minimal invariant subgroup $T_1$ of $H$ has order at most $3^s$. Now $H/C_H(T_1)$ is isomorphic to a subgroup of the group of all automorphisms of $T_1$. But $T_1$ is elementary Abelian, hence $Aut T_1 \cong GL(k, 3)$, where $k = 3$, and the order of $Aut T_1$ is not divisible by $5$. Thus, $P \subseteq C_H(T_1)$, but then $T_1 \subseteq C_H(P) = P$. Contradiction.

Assume there exists a subgroup $K$ of order $2^{5s}$. Let $L$ be a minimal invariant subgroup of $K$. Since $N_{G}(P)$ has order $2^{5s}$, it follows that $P$ is non-invariant in $K$ and that $L$ is a $2$-group. By Maschke's theorem (see [3, p. 122]), $L$ is a direct product of irreducible $P$-subgroups $L_i$. The subgroup $P$ is self-centralizing, hence the $L_i$ do not centralize $P$ and, according to [3, Theorem II.3.10], the order of $L_i$ is $2^i$ for all $i$. Consequently, $i = 1$ and $L_1 = L$. The factor group $K/L$ has order $20$, hence it is $5$-closed and $L_1P$ is invariant in $K$. Now $K = LNK(P)$. The intersection $L \cap NK(P)$ is invariant in $NK(P)$, hence $L \cap NK(P) \subseteq C_K(P) = P$. Thus, $L \cap NK(P) = 1$, and $R/L$ is isomorphic to a cyclic group of order $4$ in $NK(P)$. This contradicts the fact that $R/R'$ has exponent $2$.

If $G$ contains a subgroup of order $2^{5s^2}$, then the index of this subgroup in $G$ is $5$. Therefore, $G$ is isomorphic to a subgroup of the symmetric group $S_5$ of degree $5$. But the order of $G$ is greater than that of $S_5$. Contradiction.