A Uniqueness Result for a Nonlinear Hyperbolic Equation (*).

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Summary. — We prove the uniqueness of the solution of a Cauchy problem for a nonlinear and non-local hyperbolic equation, which arises in a model of the dynamic of cardiac muscle.

0. — Introduction.

A model of the contracting mechanism of the cardiac muscle has been introduced in the framework of a collaboration between physiologists and mathematicians of the University of Pavia (see [1], [2], [5], [6], [7]).

According to the sliding filament theory of HUXLEY (see [4]), this model leads to a nonlinear and non-local hyperbolic partial differential equation. The aim of this paper is to study the Cauchy problem for such an equation, focusing the question of the uniqueness of the solution.

The dynamic of the contraction is described by the following equation.

\[
\begin{aligned}
\frac{\partial}{\partial t} u(x, t) + \frac{d}{dt} \ln \left( \frac{\int x u(x, t) \, dx - Q + q}{1 + \int x u(x, t) \, dx} \right) \frac{\partial}{\partial x} u(x, t) &= \\
&= \gamma(t) f(x)[1 - u(x, t)] - g(x) u(x, t), \quad x \in \mathbb{R}, \ t \in [0, T] \\
\end{aligned}
\]

(0.2)

\[ u(x, 0) = \varphi(x), \quad x \in \mathbb{R}. \]

In (0.1), \((\xi)^+\) is the positive part of \(\xi\), \(u(x, t)\) is the unknown, and \(\gamma(t), f(x), g(x), \varphi(x), Q\) and \(q\) are given.

From a physiological point of view the data satisfy

(0.3) \(\gamma(t) > 0, \quad f(x) > 0, \quad g(x) > 0, \quad x \in \mathbb{R}, \ t \in [0, T]\)

(0.4) \(0 < \varphi(x) < 1, \quad x \in \mathbb{R}\)

(0.5) \(f(x)\) and \(\varphi(x)\) have compact support in \(\mathbb{R}\)

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(0.6) \[ \lim_{x \to \pm \infty} g(x) = \pm \infty \]

(0.7) \[ Q, q \in \mathbb{R} \quad \text{and} \quad Q > q > 0 . \]

The equation (0.2) takes two different experimental situations into account:

a) the so-called isometric contraction, characterized by the inequality

(0.8) \[ \int_{\mathbb{R}} xu(x, t) \, dx < Q - q ; \]

b) the so-called isotonic contraction, characterized by the inequality

(0.9) \[ \int_{\mathbb{R}} xu(x, t) \, dx > Q - q . \]

For more details on this mathematical model we refer to [1], [2], [3], [6], [7] and [9].

In general the contraction of cardiac muscle is a sequence of both isometric and isotonic contractions; hence also the free boundary of the subset of \([0, T]\) where the solution is isotonic, is an unknown of the problem.

COMINCIOLI and TORELLI proved in [2] the existence and uniqueness of a solution of the only isometric problem, that is, to find \(v(x, t)\) such that

(0.10) \[ \frac{\partial}{\partial t} v(x, t) - \frac{d}{dt} \left[ \ln \left( 1 + \int_{\mathbb{R}} xu(x, t) \, dx \right) \right] \frac{\partial}{\partial x} v(x, t) = \]

\[ = - \gamma(t)f(x)(1 - v(x, t)) - g(x)v(x, t) , \quad x \in \mathbb{R}, \ t \in [0, T] \]

(0.11) \[ v(x, 0) = 0 , \quad x \in \mathbb{R} . \]

In [9] TORELLI studied the problem (0.1), (0.2) and proved the global existence of a solution when \(\varphi(x)\) is identically 0.

The details of the proof of the global existence of a solution of (0.1)-(0.2), for any admissible initial value \(\varphi\), are contained in [3]. In this paper we prove the global uniqueness of the solution \(u(x, t)\) of (0.1), (0.2) for any \(\varphi\) satisfying (0.4) and (0.5).

We notice that the main difficulty of the problem depends on the fact that the equation (0.1) has a non-linearity which is not only non-local but also non-continuous with respect to the unknown \(u\), because of the «derivative of the positive part». Hence the usual Gronwall’s type techniques fail when

\[ \int_{\mathbb{R}} xp(x) \, dx = Q - q . \]