STUDIES IN THE APPLICATION OF RECURRENCE RELATIONS TO SPECIAL PERTURBATION METHODS

V. Reduction in the Number of Auxiliary Variables, and Automatic Step-length Adjustment by Reverse Integration, with Application to the Restricted Three-Body Problem

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Abstract. The procedure of numerical integration of the elliptic three dimensional restricted three-body problem by the use of recurrence relations to evaluate successively higher derivatives of the relative position and velocity vectors of the bodies and of the variational matrix is investigated. A set of recurrence relations is developed which involves the introduction of fewer auxiliary variables than in previous papers of this series, while the recurrence relations themselves are of a simpler form than those in other treatments involving the same number of such auxiliary variables. A technique for automatic adjustment of the integration step-length at each point in the orbit, such that the local truncation error remains close to, but always less than, some specified amount, is incorporated. This technique involves the comparison of pre-integration values with those obtained after consecutive forward and reverse integration steps, and has decided advantages over step-adjustment methods currently in use.

Both these modifications to previous techniques are shown, by presentation of sample computational results, to represent considerable savings in machine time for a given calculation and desired accuracy; these savings are generally around a factor of two and become greater as the desired accuracy in the computations increases.

1. Introduction

In a previous paper of this series (Roy et al., 1973; hereafter referred to as Paper I), it was found that, in the two-body problem, the use of recurrence relations to compute the coefficients of an explicit Taylor series integration step resulted in large savings of machine time, and also a greater accuracy, compared with classical single-step integration procedures such as Runge-Kutta Four (RK4). The use of recurrence relations as a tool in integrating various problems in celestial mechanics has been pursued by other authors (e.g., Myachin and Sizova, 1970; Broucke, 1971; Moran, 1972, 1973; Moran et al., 1973; Black, 1973).

In particular, Moran (1973; hereafter referred to as Paper II – see also Moran, 1972) applied recurrence relations to the three dimensional elliptic restricted three-body (ER3) problem in order to determine classes of orbit for which one or either of the standard Cowell or Encke formalisms of the equations of the problem (cf. Danby, 1962) was preferable. Due to the large amount of current interest in the ER3 problem and its various special cases (cf. Zagouras and Markellos, 1977), particularly with reference to periodic orbits, the present paper seeks to improve on the recurrence relation scheme of Paper II, while also presenting the variational equations for the
system – used in both finding, and determining the stability of, periodic orbits (cf. Zagouras and Markellos, 1977) – in recurrence relation form.

In Paper II, expressions for the Taylor series coefficients were derived using a set of four auxiliary variables (u, w, s and σ in the notation of Paper II) per relative position vector, each of these being represented by an infinite power series in the time t. In this way the differential equations of the system were reduced to quadratic form; substitution of the power series representation of the variables and equating coefficients of powers of t then yielded the required recurrence relations. The integration step-length at each point in the orbit was automatically adjusted throughout the integration; this was accomplished by comparing the results of a single forward integration step with those obtained from two consecutive half-steps, subsequently halving, doubling, or maintaining the same step length, as appropriate, so as to keep the local truncation error below some specified amount.

Instead of four auxiliary variables per relative vector, we herein find that we need only introduce two (u and s in the notation of Paper II). Integration of the variational equations requires the introduction of one additional auxiliary variable into both schemes. In our procedure, the recurrence relations are obtained by application of Leibniz’s Theorem rather than by equating coefficients in an infinite power series. This obviates the need for the basic differential equations of the system to be in quadratic form and so enables the above reduction in the required number of auxiliary variables to be effected. Moreover, our scheme has greater simplicity than schemes such as that of Myachin and Sizova (1970), whose treatment, although involving the same number of auxiliary variables (u and w−1 in the notation of Paper II), results in more complicated expressions for the Taylor series coefficients. Also, in the present treatment, the integration step-length is adjusted by comparing values of each position and velocity vector both before the integration step and after consecutive forward and reverse steps with the same step-length. This removes the need for wasteful evaluation of derivatives at the half-way point (for approximately half the number of steps – see Section 3) and also admits a much more accurately fitted step-length to be used throughout the orbit.

The above improvements all constitute substantial savings in machine time, and therein lies their importance. After establishing the recurrence relations in Section 2, and presenting the method of step adjustment in Section 3, we illustrate the power of the new method in Section 4, by providing computational results which compare the methods of Paper II and the present paper. In Section 5 we summarize the results obtained and present some conclusions.

2. Equations and Recurrence Relations

In the ER3 problem, particles S, of mass (1 − μ), and J, of mass μ, revolve around each other in a Keplerian ellipse, and we are concerned with the motion of a particle P, of negligible mass, in the combined gravitational field of S and J. In an inertial