Models of Relativistic Hamiltonian Interactions

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Abstract

A model of a relativistically invariant Hamiltonian 2-particle interaction is given. It is classical in the sense of having 6 degrees of freedom. This model shows that the author’s earlier general definition of such systems is not vacuous. In this model the forces of interaction die away as the particles are removed from each other.

1. Definitions

The general method for constructing such Lorentz-invariant systems of interacting particles was presented in (Arens, 1974). These systems are completely Hamiltonian (see footnote 1). Briefly, the entire class of interactions defined in (Arens, 1974) is obtained in the following way.

Say there are two particles. Let $\mathbb{R}^4$ be four-dimensional Cartesian space, regarded as space-time. Form $M \times M$ with coordinates $x^1, x^2, x^3, x^4, y^1, y^2, y^3, y^4$. Then form $T_1(M \times M)$, the cotangent bundle over $M \times M$ with coordinates $x_i, x^j, p_i, p^j, q_i, q^j$. In $T_1(M \times M)$ there is defined a symplectic structure and a Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial p_i} + \frac{\partial F}{\partial y^i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x^i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial y^i}$$

where, as henceforth, we sum on repeated indices from 1 to 4.

We now select a surface $\mathcal{S}_2$ in $T_1(M \times M)$ invariant under the standard action of the Poincaré group, having also properties H-1, H-2, H-3 as follows (footnote 2):

1. As emphasized in (Arens, 1974), these systems do not have all the properties enumerated in (Currie, Jordan, and Sudarshan, 1963).
2. We present them in a different order from, but with the same numbering as that given in (Arens, 1974). The wording in (Arens, 1974) is more complicated because invariance is not there postulated at the outset.

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(H-2). There must exist functions $H_1, H_2$ depending only on $x^1, x^2, x^3, x^4, y^1, y^2, y^3, y^4, p_1, p_2, p_3, q_1, q_2, q_3$, but defined for all values of these variables satisfying

\[(x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 > (x^4 - y^4)^2 \tag{1.1}\]

such that $J_2$ is described by

\[p_4 + H_1 = 0, \quad q_4 + H_2 = 0 \tag{1.2}\]

$H_1$ and $H_2$ are subject to the condition

\[\{p_4 + H_1, q_4 + H_2\} = 0 \tag{1.3}\]

A motion is a maximal connected integral submanifold for the singular distribution of the restriction to $S_2$ of the symplectic structure of $T_1(M \times M)$ (Arens, 1974).

These motions are 2-dimensional.

(H-3). Each pair of values $(x^4, y^4)$ occurs once and only once on each motion.

If these conditions are formulated for only one particle instead of two then (H-1) becomes vacuous, (H-2) says that there is a Hamiltonian while (H-3) says that given any initial conditions there is a motion existing for all times $x^4$.

Returning to two particles, it is shown in (Arens, 1974) that the resulting system has a Hamiltonian

\[H(x^1, x^2, x^3, y^1, y^2, y^3, p_1, p_2, p_3, q_1, q_2, q_3, t) = H_1(x^1, x^2, x^3, t, y^1, y^2, y^3, t, p_1, p_2, p_3, q_1, q_2, q_3)
+ H_2(x^1, x^2, x^3, t, y^1, y^2, y^3, t, p_1, p_2, p_3, q_1, q_2, q_3) \tag{1.4}\]

The system will have zero interaction if and only if the six functions $\{\partial H/\partial x^i, H\}$ $\{\partial H/\partial y^j, H\}$ are identically zero. It was argued in Arens (1974) that there is no formal obstacle to having a nonzero interaction. In the present paper we want to exhibit examples to show that (H-2) and (H-1) can be satisfied without having a zero interaction. Reference to our remarks about single particles shows that (H-3) is a still more difficult matter. Since we do not exhibit our $H_1$ and $H_2$ explicitly, it is difficult to establish the required Lipschitz conditions for our examples. At any rate, we have not established (H-3) for our examples.

2. Defining an Invariant $S_2$

We will insure the invariance of $J_2$ under the Poincaré group by defining it by equations

\[F_1 = 0, F_2 = 0 \tag{2.1}\]

where $F_1$ and $F_2$ are functions on $T_1(M \times M)$ which are invariant under the group. (We are then still permitted to discard components of the surface defined by 2.1.)