Estimation of a Displacement Parameter of a Quantum System

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Abstract
A displacement parameter such as the angle of rotation or the position of a quantum system, or the phase of a harmonic oscillation, is to be estimated by observing the system with an apparatus that applies to it an operator-valued measure (o.v.m.). The o.v.m. minimizing the average cost of errors in the estimate is determined by quantum estimation theory for a system in a pure state. The best estimate of the parameter is found to be the more accurate, the greater the uncertainty of the dynamical variable serving as the infinitesimal generator of the displacement group. The relation of this result to such uncertainty principles as those between angle and angular momentum, and between oscillator phase and photon number, is discussed. A lower bound to the variance of an unbiased estimate of the time of occurrence of an event in the evolution of a system is derived from the quantum-mechanical Cramér-Rao inequality. It is inversely proportional to the square of the uncertainty of the energy of the system.

1. Quantum Estimation Theory

The uncertainty principle for two quantum-mechanical variables, represented by Hermitian operators $\mathcal{A}$ and $\mathcal{B}$, is expressed by the formula

$$\Delta \mathcal{A}^2 \Delta \mathcal{B}^2 \geq \frac{1}{4} \langle \mathcal{F} \rangle^2$$

(1.1)

in which $\mathcal{F} = [\mathcal{A}, \mathcal{B}]$ is the commutator of $\mathcal{A}$ and $\mathcal{B}$, and $\langle \mathcal{F} \rangle$ denotes the expected value (Robertson, 1929). The uncertainties $\Delta \mathcal{A}$ and $\Delta \mathcal{B}$ are defined by

$$\Delta \mathcal{A}^2 = \langle (\mathcal{A} - \langle \mathcal{A} \rangle)^2 \rangle, \quad \Delta \mathcal{B}^2 = \langle (\mathcal{B} - \langle \mathcal{B} \rangle)^2 \rangle$$

(1.2)

The most familiar such relation connects the uncertainties of the position $\mathcal{P}$ and the momentum $\mathcal{P}$ of a particle.

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If one variable is the angular position or azimuth $\theta$ of a quantum system about an axis, and the other is the component $J_z$ of angular momentum along that axis, one expects an uncertainty relation of the form
\[ \Delta J_z \Delta \theta \geq \hbar \] (1.3)
for like $\mathcal{P}$ and $\mathcal{Z}$ these are conjugate variables in classical mechanics; but as indicated by Judge (1964), an operator corresponding to the azimuth $\theta$ and possessing with $J_z$ a commutator of the necessary form cannot be defined. The supposed uncertainty relation
\[ \Delta n \Delta \theta \geq \hbar \] (1.4)
between the number $n$ of photons in an harmonic oscillator and the phase $\theta$ of its oscillation suffers a similar impediment. These difficulties have been circumvented by introducing operators $C$ and $S$ corresponding to $\cos \theta$ and $\sin \theta$ and by establishing for them a more complicated uncertainty relation that goes into (1.3) and (1.4) in the limit of large quantum numbers. This approach has been thoroughly reviewed by Carruthers & Nieto (1968). Here we shall describe how a more general formulation of quantum measurement, when utilized in quantum estimation theory, permits a simpler derivation and interpretation of uncertainty relations involving such variables as an azimuth or phase. These are viewed not as dynamical variables represented by operators, but as parameters of the state of the system.

Suppose that an apparatus $A_P$ prepares a quantum system $S$ in a state $|\psi\rangle$. If the combination of apparatus and system is rotated through an angle $\theta$ about the $z$-axis, the apparatus $A_P$ will instead prepare the system $S$ in the state
\[ |\psi(\theta)\rangle = e^{iN\theta} |\psi\rangle \] (1.5)
where $N$, the infinitesimal generator of the group of rotations about the $z$-axis, is the operator $J_z/h$, $\hbar = \text{Planck's constant } h/2\pi$. In a Stern-Gerlach experiment, for instance, a beam of spin $-\frac{1}{2}$ particles is divided into two beams, in one of which the spins point upward and in the other downward. The 'spin-up' beam passes through a hole in a screen, which intercepts the 'spin-down' beam. The whole device is rotated about an axis coincident with the 'spin-up' beam, which then contains particles whose spin vector points in the direction $\theta$ with respect to a $y$-axis normal to the beam.

In this way the angle $\theta$ of rotation is a parameter of the state of the system $S$, specifying its orientation about the $z$-axis. If you did know through what angle $\theta$ the preparing apparatus had turned, you might ask how you could observe $S$ in order to determine $\theta$ as accurately as possible. The results of your observations and your subsequent calculations with them would yield for the parameter $\theta$ an estimate $\hat{\theta}$ that in general would be somewhat in error, $\hat{\theta} \neq \theta$. We shall see that the azimuth $\theta$ can be the more accurately estimated, the more broadly the state $|\psi(\theta)\rangle$ is distributed among the eigenstates of the angular momentum $J_z$. 