Thermodynamic Arrow for a Mixing System

L. S. SCHULMAN and R. SHTOKHAMER

Physics Department, Indiana University, Bloomington, Indiana 47401, and
Physics Department, Technion, Haifa, Israel

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Abstract

The purpose of this paper is to study the appearance of time asymmetry in dynamical systems. The systems are harmonic oscillators and a certain mixing flow on the torus. The asymmetry is a kind of frictional force, but we emphasize that the boundary conditions, a usual source of asymmetry in studies of this sort, are taken to be time symmetric. For the mixing flow the response of the system, as reflected in its entropy as a function of time, occurs only subsequent to the "friction," while for the oscillators the effects are both before and after. Some general discussion also takes up the question of which of the foregoing systems is a better model of the physical world for purposes of correlating arrows of time.

1. Introduction

In an earlier paper (Schulman, 1973) one of us examined a thought experiment of Gold (1962a, b) which was supposed to correlate the thermodynamic arrow of time with the cosmological arrow. According to Schulman (1973), a correct framework for demonstrating such a correlation is the following: For some model dynamical system sufficient (but not excessive) data about the state of the system are given at two different times (t = 0 and t = T_1 + T_2), and the state is studied at intervening times. At some intermediate time T_1 a time reversal noninvariant perturbation is applied to the system. If the data at 0 and T_1 + T_2 are roughly the "same" (in what sense will become evident), then the only asymmetry (or arrow) in the problem is the perturbation at T_1. To demonstrate a correlation of this arrow with some other arrow, one would

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2 If one agrees with E. R. Harrison (1974) it would be more accurate to say that Gold uses a "dark sky arrow" rather than an arrow determined by the expansion of the universe (Gold uses the fact that more photons leave than arrive). In any case, we continue to use the terminology "cosmological arrow."

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then have to show the emergence of the other arrow in the behavior of the system. In particular, it would be interesting to see whether the response (by the system) to the perturbation occurs both before and after the perturbation or only on one side.\(^3\)

In Schulman (1973) a collection of harmonic oscillators was studied and no arrow found. That is, boundary values for the oscillators were given at \(t = \pm T_0\) and friction allowed to act at \(t = 0\). By considering oscillators with many different frequencies a sort of equilibrium was reached for \(-T_0 + \tau < t < -\tau\) and \(\tau < t < T_0 - \tau\), where \(\tau\) is some characteristic equilibration time. However, the system was found to depart from equilibrium both for \(0 < t < \tau\) and \(-\tau < t < 0\). Arguments were also given for this noncausal behavior to occur in dynamical systems more general than harmonic oscillators.

In this paper we study another model, but are able to report that this system does show causal behavior. The model is a certain flow on a torus (details below) and was selected because the flow is known to be mixing. It was felt that the mixing property would facilitate the appearance of phenomena associated with time asymmetry, and this indeed appears to be the case. A perturbation of a time asymmetric type is applied and the system found to be out of equilibrium only after the perturbation.

Section 4 contains some general arguments which explore the question of which model is a better description of natural phenomena, the oscillators or the flow on the torus, with the object of determining whether optimism or pessimism is in order for the program of correlating arrows.

2. Dynamical Systems

Two dynamical systems will be considered: harmonic oscillators and discrete flow on a torus.

A harmonic oscillator with natural frequency \(\omega\) has Hamiltonian

\[
H = \frac{1}{2}(p^2 + \omega^2q^2)
\]

where \(p\) and \(q\) are Cartesian coordinates in its phase space. In polar coordinates,

\[
E = \frac{1}{2}(p^2 + \omega^2q^2)
\]

\[
\theta = \tan^{-1}(\omega q/p)
\]

\(^3\) In Gold's thought experiment the system is a star confined to a box for millions or perhaps billions of years. At some stage a window is opened and photons escape. The expansion of the universe and darkness of the night sky are what cause this asymmetric behavior, namely, the net escape rather than the absorption of photons. The question for this system is whether it departs from equilibrium only after the opening of the window, or also before. See Schulman (1973) and Gold (1962a, b).