Parastatistics and the Lie Algebraical Methods Used There

T. PALEV†
Cern, Geneva

Received: 1 August 1973

Abstract

We review some of the properties of the parafield operators and discuss in some detail where the difference between the ordinary and paraquantisation originates. Particular attention is paid to the many-vacuum representations of the para-Fermi operators.

1. Introduction

In this paper I would like to mention some properties of the parastatistics and in particular to draw the readers attention to the very close connection between the representations of the para-Fermi operators and the algebra of the orthogonal group.

To begin with I must mention that parastatistics was introduced by Green (1953) who observed that the commonly accepted rules of second quantisation, although sufficient, are not necessary however for satisfying all physical requirements, and pointed out how the quantisation axioms can be generalised. It is worth considering here in more detail the question of where the generalisation comes from. For simplicity, let us consider the quantisation on an example of a real scalar free field \( \phi(x) \).

2. Classical Case

One starts with the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi(x) \cdot \partial^{\mu} \phi(x) - \frac{1}{2} m^2 \phi^2(x)
\]  

(2.1)

† On leave of absence from the Institute for Nuclear Research and Nuclear Energy, Boul. Lenin 72, Sofia 13, Bulgaria (address after September 3, 1973).

Copyright © 1974 Plenum Publishing Company Limited. No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of Plenum Publishing Company Limited.
The Lagrange-Euler equation in this case is the Klein-Gordon equation with positive and negative frequency solutions \( \varphi^\pm(x) \) of the form:

\[
\varphi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\kappa}{\sqrt{(2\kappa^0)}} e^{\pm i\kappa x} \varphi^\pm(\kappa), \quad \kappa^0 = \sqrt{\kappa^2 + m^2}
\]  

(2.2)

Noether's theorem, together with the invariance of \( S \) under the Poincaré group \( \mathcal{P} \), gives the invariant quantities: the energy-momentum vector \( P^n \) and the angular-momentum tensor \( M^m_n \). In particular, \( P^n \) may be represented as

\[
P^n = \frac{1}{2} \int dp_p^n [\varphi^+(p), \varphi^-(p)]_+
\]

(2.3)

3. Quantisation of the Field

The quantisation can be performed in different equivalent ways; for instance, by postulating the equal-time commutation relations. For what we want to show, however, it is more convenient to follow the quantisation procedure given by Bogoliubov & Shirkov (1959). It is based on the following postulates:

1. the field \( \varphi(x) \) becomes an operator;
2. the energy-momentum vector \( P \) and the angular-momentum tensor \( M \) are expressed in terms of the operator-field functions by the same expressions of the type (2.3), i.e., as in the classical case, with proper ordering of the operator factors.

It follows now from (1) and (2), together with the requirements that the field transforms according to a unitary representation of \( \mathcal{P} \) and the compatibility of the transformation properties of the field and the state vectors, that \( \varphi(x) \) satisfies the commutation relation

\[
[P^n, \varphi^\pm(\kappa)] = \pm \kappa^n \varphi^\pm(\kappa)
\]

(3.1)

Inserting (2.3) and (3.1), we have

\[
[[\varphi^+(p), \varphi^-(p)], \varphi^\pm(\kappa)] = \pm \delta(p - \kappa) \varphi^\pm(\kappa)
\]

(3.2)

Generalising this equation, Green has postulated the three-linear structure relation for the parafields

\[
[[\varphi^+(q), \varphi^+(p)], \varphi^\pm(\kappa)] = 0
\]

\[
[[\varphi^+(q), \varphi^-(p)], \varphi^\pm(\kappa)] = \pm \delta(p - \kappa) \varphi^\pm(\kappa)
\]

(3.3)

where the anticommutator (commutator) corresponds to the para-Bose (para-Fermi) field.

In the ordinary quantisation, one has to add two more postulates:

3. the commutator or the anticommutator of two-field operators is a \( C \) number;