Summary. – A genuine faceless cone is a non-empty linear cone that is open in some linear topology and includes no line. This paper describes all assignments of metrics to genuine faceless cones such that every linear mapping between cones is a contraction.

1. – Introduction.

In an earlier paper [2], W. NOLL and the present author investigated a "natural" metric—the "distance-function" δ—on each linear cone of a certain kind, and the properties of mappings between such cones that are isometric with respect to the respective distance-functions. A special case of this "natural" metric had been introduced by Noll in [1; p. 291] for the cone of strictly positive symmetric bilinear forms on a three-dimensional real linear space.

In this paper we intend to examine in what sense this distance-function δ is "natural". To speak somewhat vaguely, we are looking for a metric to be assigned to each cone under consideration, with the requirement that this assignment share significant properties with the assignment of the distance-functions defined in [2]: more specifically, we require that each linear mapping from cone to cone be a contraction with respect to the metrics assigned to the domain and the codomain of the mapping. We shall show in Section 3 that this requirement places very strong restrictions on the assigned metrics, though not quite strong enough to single out the distance-functions (together with their obvious variants, obtained by composing with some fixed isotone subadditive function). The classification of these assignments of metrics depends on certain functions of two real variables, the "adequate functions" that we discuss in Section 2.

The linear cones that we shall consider are genuine faceless cones, as defined in [2]. In brief, a subset C of a real linear space is a (linear) cone if it is stable under addition and under multiplication by strictly positive numbers, i.e., $C + C \subseteq C$ and $\mathbb{P} \cdot C \subseteq C$. ($\mathbb{R}$ will denote the set of real numbers, and $\mathbb{P}$ and $\mathbb{P}^\circ$ the subsets of all positive and all strictly positive ones, respectively.) If C is not empty, its linear span is $\text{Sp } C = C - C$. The cone C is genuine if it is not empty, does not contain 0,
and includes no line (1-dimensional affine subspace); it is *faceless* if for every \( u, v \in C \) there is \( \lambda \in \mathbb{P}^\infty \) such that \( \lambda v - u \in C \) (cf. [2; Proposition 2.5]). It may be helpful to mention that a cone \( C \) is faceless if and only if it is open in some linear topology on its linear span \( \text{Sp} \ C \) ([2; Proposition 6.2]).

We denote the particular genuine faceless cone \((\mathbb{P}^\infty)^2\) by \( K \).

If \( C \) is a genuine faceless cone, we may define its *gauge-function* \( \kappa : C^2 \to \mathbb{P}^\infty \) by

\[
\kappa(u, v) := \inf \{ \lambda \in \mathbb{P}^\infty | \lambda v - u \in C \}, \quad \text{for all } u, v \in C.
\]

When specification of the cone \( C \) is essential, we write, in full, \( \kappa_C \) instead of \( \kappa \). The properties of this function are discussed at length in [2; Section 4]; some of them are recited at the end of this introduction. The *distance-function* \( \delta \) of \( C \), denoted in full by \( \delta_C \), is defined by

\[
(1.1) \quad \delta(u, v) := \log \max \{ \kappa(u, v), \kappa(v, u) \}, \quad \text{for all } u, v \in C.
\]

One of the conclusions in Section 3 will be that each one of the *natural* metrics that we are considering depends on its arguments only through the values of the gauge-function, just as does the distance-function.

In [2] the nature of those mappings between genuine faceless cones that are isometric bijections with respect to the distance-functions was analysed with some care. In Section 4 we shall characterize the mappings that are isometric with respect to *every one* of the *natural* assignments of metrics (or at least with respect to two suitably chosen assignments) as being precisely those that preserve the gauge function together with those that replace it by the gauge-function with the arguments interchanged (*gauge-preserving* and *gauge-reversing* mappings).

We shall need the following properties of the gauge-function \( \kappa \) of a genuine faceless cone \( C \):

\[
(1.2) \quad \kappa(u, v) \kappa(v, u) \geq \kappa(u, u) = 1, \quad \text{for all } u, v \in C,
\]

\[
(1.3) \quad \kappa(u, v) \kappa(v, w) \geq \kappa(u, w), \quad \text{for all } u, v, w \in C,
\]

\[
(1.4) \quad \kappa(\alpha u, \beta v) = (\alpha/\beta) \kappa(u, v), \quad \text{for all } u, v \in C \text{ and } \alpha, \beta \in \mathbb{P}^\infty,
\]

\[
(1.5) \quad \text{For all } u, v \in C, \quad \kappa(u, v) = \kappa(v, u) = 1 \text{ if and only if } u = v,
\]

\[
(1.6) \quad \text{For all } u, v \in C, \quad \kappa(u, v) \kappa(v, u) = 1 \text{ if and only if } u \text{ and } v \text{ are collinear.}
\]

Formulas (1.2), (1.3), (1.4) are immediate consequences of the definitions; (1.5) follows from [2; Proposition 4.1] and the fact that the cone \( C \) is genuine; (1.6) follows from (1.5) and (1.4).