Reliability-based optimum design of a symmetric laminated plate subject to buckling

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Abstract This study is concerned with the buckling reliability maximization of a symmetric laminated composite plate with respect to the mean ply orientation angle. The reliability is evaluated by modeling the buckling failure as a series system consisting of potential eigenmodes. The mode reliability is obtained by the first-order reliability theory (FORM), where material constants and orientation angles of individual layers, as well as the applied loads, are treated as random variables. In order to keep track of the intended buckling mode during the reliability analysis, the mode tracking method is utilized. Then, the failure probability of the series system is approximated by Ditlevsen's upper bound. The reliability maximization problem is formulated as a nested problem with two levels of optimization. Through numerical calculations, the reliability-based design is demonstrated to be important for the structural safety in comparison with the deterministic buckling load maximization design.

1 Introduction
Laminated composite plates are widely used in structural applications because of their high specific strength and stiffness. As in the case of any plate, the presence of in-plane loads may cause buckling. Therefore, many studies have been conducted on the buckling load maximization design of composite plates (Haftka and Giurdal 1992).

However, most of them yield the optimal fiber orientation angles under deterministic conditions, where the material properties and the load conditions are assumed to have no variations. It has been known that such a deterministic optimal design is strongly anisotropic and sensitive to the change in loading conditions (Haftka and Giurdal 1992; Park 1992). Therefore, it is necessary to consider the effect of such variations in loadings and material properties by applying the structural reliability theory (Thoht-Christensen and Murotsu 1986). Especially, the reliability-based design which maximizes the structural reliability is important (Thoht-Christensen and Murotsu 1986; Sørensen 1986; Enevoldsen 1991).

The reliability analysis and the reliability-based design under the inplane strength by the first ply failure criterion have already been studied (Miki et al. 1990, 1992; Shao et al. 1993; Murotsu et al. 1994). The studies have shown that the reliability increases as the number of fiber axes is increased, and that the reliability-based design approaches a quasi-isotropic laminate construction. The design is very different from the deterministic optimum design of which the orientation angle runs along the loading direction. However, the effect of the random variations of design parameters such as orientation angles of the composite laminated plate on the buckling load has not yet been investigated.

In this study, the buckling reliability is maximized for the simply supported symmetric laminated composite plate with respect to the mean fiber orientation angle.

2 Buckling analysis
In practical structural applications, a laminated plate is mainly stacked symmetrically with respect to the mid-plane to avoid bending-extensional coupling. However, the actual stacking sequence of the laminated plate will not be symmetric due to the random variations of material properties and orientation angle of each ply, even though the mean stacking sequence is assumed to be symmetric. Consequently, buckling analysis for an unsymmetric laminate is required.

2.1 Constitutive equations
For a general laminated plate as shown in Fig. 1a, the constitutive equations are given as follows (Whitney 1987):

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} = \begin{bmatrix}
A & B \\
B & D
\end{bmatrix} \begin{bmatrix}
\varepsilon^0 \\
\kappa
\end{bmatrix}.
\]

(1)

The A and D matrices are extensional and flexural stiffness matrices, respectively. The A matrix relates the in-plane stress resultants (N) to the mid-plane strains (\(\varepsilon^0\)) while the D matrix does the moment resultants (M) to the curvature (\(\kappa\)). The B matrix, on the other hand, relates the in-plane stress resultants to the curvature and the moment resultants to the mid-plane strains, and hence is called the bending-extensional coupling matrix.

The elements of each matrix are defined as follows:

\[
\begin{align*}
(A_{ij}, B_{ij}, D_{ij}) &= \int_{-h/2}^{h/2} \tilde{Q}_{ij}(1, z, z^2) dz \quad (i, j = 1, 2, 6), \\
B_{ij} &= \frac{1}{2} \sin 2\theta, \\
D_{ij} &= \cos 2\theta.
\end{align*}
\]

(2)

where \(h\) is the plate thickness and \(\tilde{Q}_{ij}\) is the stiffness of each ply. When the material principal axis of the layer (1 - 2) is rotated to the reference axis (x - y) as shown in Fig. 1b, the stiffnesses \(Q_{ij}\) are related to the material invariants \(U_i\) and the orientation angle \(\theta\) as follows:

\[
\begin{align*}
Q_{11} &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta, \\
Q_{12} &= U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta, \\
Q_{16} &= (U_2/2) \sin 2\theta + U_3 \sin 4\theta, \\
Q_{26} &= (U_2/2) \sin 2\theta - U_3 \sin 4\theta.
\end{align*}
\]

(3)
Fig. 1. Laminated composite plate. (a) Plate model, (b) coordinate system

where material invariants $U_i$ ($i = 1, \ldots, 5$) are given in terms of the stiffness of the unidirectional material

$$U_1 = \frac{3Q_{11} + 3Q_{22} + 3Q_{12} + 4Q_{66}}{8},$$
$$U_2 = \frac{(Q_{11} - Q_{22})}{2},$$
$$U_3 = \frac{(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})}{8},$$
$$U_4 = \frac{(Q_{11} + Q_{22} + 3Q_{12} - 4Q_{66})}{8},$$
$$U_5 = \frac{(Q_{11} + Q_{22} - Q_{12} + 4Q_{66})}{8}.$$

The stiffnesses of the unidirectional material ($Q_{11}, Q_{22}, Q_{12}, Q_{66}$) are given in terms of four independent engineering material constants

$$Q_{11} = \frac{E_1}{1-\nu_1^2}, \quad Q_{22} = \frac{E_2}{1-\nu_1^2}, \quad Q_{66} = E_6,$$

where $E_1, E_2, \nu_1$ and $E_6$ denote the Young’s moduli in the fiber direction 1 and lateral direction 2, Poisson’s ratio and shear modulus, respectively.

2.2 Reduced bending stiffness method

Since the unsymmetric laminate has bending-extensional coupling, it is difficult to directly solve the governing equation. Therefore, a reduced bending stiffness method is used here as an approximation method (Ashton 1969). The method reduces the problem to an equivalent anisotropic bending problem without bending-extensional coupling. Then, Galerkin’s method for the analysis of symmetric laminate plate can be directly utilized (Whitney 1987).

By using the constitutive relation of the plate in the form of (6), the strain energy $U$ of a laminated rectangular plate can be expressed in the following form:

$$U = \frac{1}{2} \int_{0}^{b} \int_{0}^{a} (e^{0T}N + \kappa^{T}M) \, dx \, dy =$$

$$\frac{1}{2} \int_{0}^{b} \int_{0}^{a} (N^{T}A^{*}N + \kappa^{T}D^{*}\kappa) \, dx \, dy,$$

where $a$ and $b$ are the plate dimensions of $x$ and $y$ directions, respectively. There is no bending-extensional coupling appearing in this representation. This suggests that the unsymmetric laminated plate problem can be solved approximately in the following uncoupled form:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D^{*} \end{bmatrix} \begin{bmatrix} e^{0} \\ \kappa \end{bmatrix}.$$  

Thus, the governing equation of a symmetric laminated plate can be utilized by substituting $D^{*}_{ij}$ for the bending stiffness $D_{ij}$.

2.3 Galerkin’s method

Consider a simply supported rectangular laminated plate with dimensions $a$ and $b$ subjected to uniform biaxial compression and shear loads $(N_{x}, N_{y}, N_{xy})$, as shown in Fig. 1a. The following set of algebraic equations is obtained by Galerkin’s method (Whitney 1987):

$$\pi^4 \left[ D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2R^2 + D_{22}n^4R^4 - N_{x} \frac{m^2a^2}{\pi^2} - N_{y} \frac{n^2R^2a^2}{\pi^2} \right] A_{mn} =$$

$$32\pi^2mnR \sum_{i=1}^{M} \sum_{j=1}^{N} M_{ij} \left[(m^2 + i^2)D_{16} + (n^2 + j^2)D_{26}R^2 + N_{xy} \frac{a^2}{\pi^2} \right] A_{ij} = 0, \quad \{ m = 1, \ldots, M \},$$

$$\{ n = 1, \ldots, N \}.$$  

(9)

where $R$ is the aspect ratio $a/b$ and

$$M_{ij} = \begin{cases} \pi^2 \frac{m}{a} \left[(m^2 - i^2)(n^2 - j^2) \right] & \text{if } m + i \text{ is odd} \\
\pi^2 \frac{n}{b} \left[(m^2 - i^2)(n^2 - j^2) \right] & \text{if } n + j \text{ is odd} \\
0 & \text{otherwise} \end{cases}.$$  

(10)

Equation (9) yields a set of $M \times N$ homogeneous equations. This set of equations can be reduced to an eigenvalue equation for $A_{mn}$, when the stiffness terms and the loading terms are separated.

The eigenvalue equation is written as follows:

$$[K - \lambda K_{G}] \phi = 0,$$

where $\phi$ is a vector consisting of $A_{mn}$, $K$ is the stiffness matrix and $K_{G}$ is the geometry stiffness matrix. The elements of each matrix are written as follows:

$$K_{mn} = \pi^4 \left[ D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2R^2 + D_{22}n^4R^4 \right],$$

$$K_{ij}^{mn} = -32\pi^2mnRM_{ij} \left[(m^2 + i^2)D_{16} + (n^2 + j^2)D_{26}R^2 \right],$$

$$K_{Gmn} = \pi^2a^2 \left(N_{x} + n^2R^2N_{y} \right).$$

(12)

(13)

(14)