Quasi-Residual Designs

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Dedicated to A. Ostrowski on His 75th Birthday

1. Introduction

We may partially specify a balanced incomplete block design by naming its parameter set \((v, b, r, k, \lambda)\); such a design is an arrangement of \(v\) varieties into \(b\) blocks such that each block contains \(k\) distinct varieties, each variety occurs in \(r\) blocks, each pair of varieties occurs in \(\lambda\) blocks. It is well-known (see, for example, Bose [3] or Ryscer [8]) that

\[
bk = rv,
\]

\[
\lambda(v-1) = r(k-1).
\]

Also [see Fisher (5)], one has the inequality

\[
b \geq v.
\]

If a given block is repeated \(x\) times (that is, there is a set of \(x+1\) identical blocks), then Stanton and Sprott [11] have strengthened the Fisher inequality to give

\[
b \geq (x+1)v - (x-1).
\]

In particular, it is implicit from (4) that, if \(b < 2v\), then there are no repeated blocks.

If \(b = v\) (and \(r = k\)), we speak of a symmetrical balanced incomplete block design or a \(v-k-\lambda\) system (see Mann [7], Ryscer [8], Stanton [9]). From such a design, one can always form a residual design by deleting a single block and the varieties in it; this residual design has parameter set

\[
(V, B, R, K, A),
\]

where \(V = v-k\), \(B = v-1\), \(R = k\), \(K = k-\lambda\), \(A = \lambda\).

We shall speak of a quasi-residual design as one which has the parameters of a residual design. It was first observed by Bhattacharyya [2] that there exist quasi-residual designs which are not residual designs; his example is readily available in Hall [6]. It is the purpose of this paper to discuss quasi-residual designs, using the results of Stanton-Sprott [11] and Stanton-Mullin [10] to specify structural properties of such designs.

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2. Block Intersections in Quasi-Residual Designs

Let $B_1$ be a fixed block, and let $a_i$ denote the frequency with which the integer $i$ appears among the block-intersection numbers (that is, the number of elements in $B_1 \cap B_j$). Then the numbers $a_i$ satisfy the following relations (we use the form given in (11)).

\[ \sum_{i=0}^{K} a_i = B - 1 \]  
\[ \sum_{i=0}^{K} i a_i = K (R - 1) \]  
\[ \sum_{i=0}^{K} i^2 a_i = K (K \Lambda - K - \Lambda + R) \]  

In a quasi-residual design, (7) simplifies to

\[ \sum_{i=0}^{K} i^2 a_i = K^2 \Lambda. \]  

We note that the original symmetric design has parameters

\[ (v, v, k, k, \lambda), \]

where \( v = 1 + k(k-1)/\lambda \). Hence

\[ V = (k^2 - \lambda k - k + \lambda)/\lambda, \quad B = k(k-1)/\lambda, \]
\[ R = k, \quad K = k - \lambda, \quad \Lambda = \lambda. \]

This parameter set can be simplified to

\[ V = K (K + \lambda - 1)/\lambda, \quad B = (K + \lambda) (K + \lambda - 1)/\lambda, \]
\[ R = K + \lambda, K, \lambda, \]

and equations (5), (6), and (8) become

\[ a_0 + a_1 + a_2 + \cdots + a_K = (K^2 + \lambda^2 + 2 K \lambda - K - 2 \lambda)/\lambda, \]  
\[ a_1 + 2 a_2 + \cdots + K a_K = K (K + \lambda - 1), \]  
\[ a_1 + 4 a_2 + \cdots + K^2 a_K = K^2 \lambda. \]

Multiply these equations by $\lambda(\lambda-1)$, $-(2\lambda-1)$, 1, respectively, and add. (These multipliers are chosen so as to eliminate the terms in $a_0$ and $a_{\lambda-1}$). Then a little algebra produces

\[ \sum_{i=0}^{K} [i - \lambda] [i - (\lambda - 1)] a_i = \lambda(\lambda - 1)(\lambda - 2). \]  

Equation (12) is a fundamental result from which a number of other results follow.