DIVERGENCE OF INTERPOLATION PROCESSES
ON SETS OF THE SECOND CATEGORY

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C([0, 1]) is the space of real continuous functions \( f(x) \) on \([0, 1]\) and \( \omega(\delta) \) is a majorant of the modulus of continuity \( \omega(f, \delta) \), satisfying the condition \( \lim_{\delta \to 0} \frac{\omega(\delta)}{\delta} = \infty \). A solution is given to a problem of S. B. Stechkin: for any matrix \( \mathbb{M} \) of interpolation points there exists an \( f(\omega) \in C([0, 1]), \omega(f, \delta) = \omega(\delta) \) whose Lagrange interpolation process diverges on a set \( \mathbb{A} \) of second category on \([0, 1]\).

Let \( \mathbb{M} = \{x_{k,n}\}, 0 \leq x_{1,n} < x_{2,n} < \ldots < x_{n,n} \leq 1, \) \( n = 1, 2, 3, \ldots \), be a matrix of interpolation points belonging to the segment \([0, 1]\). Then for any real continuous function \( f(x) \) on the segment \([0, 1]\), \( f(x) \in C([0, 1]) \), we set

\[
L_n(\mathbb{M}, f, x) = \sum_{k=1}^{n} f(x_{k,n}) l_{k,n}(\mathbb{M}, x), \quad n = 1, 2, 3, \ldots,
\]

where

\[
l_{k,n}(\mathbb{M}, x) = \frac{\omega_n(x)}{\omega_n(x_{k,n}) (x - x_{k,n})}, \quad \omega_n(x) = \prod_{i=1}^{n} (x_i - x_{i,n}).
\]

We denote by \( \omega(f, \delta) \) the modulus of continuity of the function \( f(x) \in C([0, 1]) \), and by \( \Omega_0 \) the set of all real, semiadditive, continuous, nondecreasing functions \( \omega(\delta) \) on the segment \([0, 1]\), such that

\[
\omega(0) = 0 \quad \text{and} \quad \lim_{\delta \to 0^+} \frac{\delta}{\omega(\delta)} = 0.
\]

If \( \omega(\delta) \in \Omega_0 \) and \( \omega(f, \delta) = o(\omega(\delta)) \), then we say that \( f(x) \in C^*(\omega, [0, 1]) \).

In [1] it was shown that if \( \omega(\delta) \in \Omega_0 \) and

\[
\lim_{n \to \infty} \omega(1/n) \ln n = \infty, \quad (1)
\]

then for any matrix \( \mathbb{M} = [0, 1] \) there is a point \( x_0 \in [0, 1] \) and a function \( f(x) \in C^*(\omega, [0, 1]) \), such that

\[
\lim_{n \to \infty} |L_n(\mathbb{M}, f, x_0)| = \infty. \quad (2)
\]

Sergei Borisovich Stechkin posed this problem in a personal conversation: prove for any matrix \( \mathbb{M} \) the existence of a function \( f(x) \in C([0, 1]) \), whose Lagrange interpolation process diverges on a set \( \mathbb{G} \subset [0, 1] \) of second category.

The goal of the present note is the proof of the following theorem.


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THEOREM. Let the function \( \omega(\delta) \in \Omega_0 \) satisfy (1) and let \( M \) be an arbitrary matrix. Then there exists a function \( f(x) \in C_* (\omega, [0, 1]) \) and a set \( \delta \subset [0, 1] \) of the second category such that (2) is true everywhere in \( \delta \).

In what follows, all arguments are carried out for an arbitrary fixed matrix \( M \) and given function \( \omega(\delta) \in \Omega_0 \) satisfying (1).

**LEMMA 1.** There exists a countable set \( \{x_i\} \), dense in the segment \( [0, 1] \), a sequence \( \{f_i(x)\} \) of functions \( f_i(x) \in C_* (\omega, [0, 1]) \) and a nonnegative function \( \beta(x) \in C([0, 1]) \) such that

1. the function \( \beta(x) \) depends only on \( M \) and \( \omega \) and

\[
\lim_{x \to \delta} \beta(x) = 0; \tag{3}
\]

2. for any \( i = 1, 2, 3, \ldots \) one has

\[
\max_{x \in [0, 1]} |f_i(x)| \leq \beta(\delta) \omega(\delta), \quad 0 \leq \delta \leq 1, \tag{4}
\]

\[
\lim_{n \to \infty} |L_n (M, f_i, x_i)| = \infty. \tag{5}
\]

The existence of a set \( \{x_i\} \) dense in \([0, 1]\) and of a sequence \( \{f_i(x)\} \) of functions \( f_i(x) \in C_* (\omega, [0, 1]) \) for which (6) holds follows from the remark on p. 286 in [1] about the theorem of [1], and the existence of a function \( \beta(x) \) satisfying (3)-(5) follows from the method of proof of the theorem of [1]. The lemma is proved.

Let the function \( \beta(x) \) be from Lemma 1 and let \( M \) be the set of functions \( f(x) \in C_* (\omega, [0, 1]) \) for which (4) and (5) are true. Then, arguing as in the proof of the theorem of Arzela-Ascoli [2], it is easy to see that the set \( M \) is compact in \( C_* (\omega, [0, 1]) \). It is obvious that \( C** (\omega, [0, 1]) \) is a convex, compact, complete metric space, and consequently, by the Baire theorem (see [3] or [2]) is a set of the second category in itself. In addition, by virtue of (4) and (5), we have \( C** (\omega, [0, 1]) \subset C_* (\omega, [0, 1]) \).

**LEMMA 2.** There exists a countable set \( \{x_i\} \) dense in the segment \([0, 1]\), and a function \( f(x) \in C** (\omega, [0, 1]) \) such that,

\[
\lim_{n \to \infty} |L_n (M, f, x_i)| = \infty, \quad i = 1, 2, 3, \ldots \tag{6}
\]

Proof.† By virtue of Lemma 1 there exists a set \( \{x_i\} \), dense in \([0, 1]\), and a sequence \( \{f_i(x)\} \) of functions \( f_i(x) \in C** (\omega, [0, 1]) \) such that (6) is true. We choose a point \( x_j \in \{x_i\} \) and let \( F_q \) be the set of those functions \( f(x) \in C** (\omega, [0, 1]) \) for which one has

\[
\sup_{n} |L_n (M, f, x_j)| \leq q.
\]

Since all the functionals \( L_n (M, f, x_j) \) are continuous, the set \( F_q \) is closed. We set

\[
X = \bigcup_{q=1}^{\infty} F_q
\]

and we shall show that the set \( Y \) is of the first category in \( C** (\omega, [0, 1]) \).

If \( X \) were of the second category in \( C** (\omega, [0, 1]) \), then there would exist a set \( F_{q_0} \) which would not be nowhere dense in \( C** (\omega, [0, 1]) \). Let \( F_{q_0} \) be dense in some ball \( K \) of radius \( \rho, \rho > 0 \). Since the set \( F_{q_0} \) is closed, \( F_{q_0} \supset K \), which means that for any function \( f(x) \in K \)

† The space \( C** (\omega, [0, 1]) \) is not a B-space, and hence we do not apply the principle of condensation of singularities (see [2]), nevertheless we owe the idea of the proof of the lemma to Theorem 1.5.1. of [4].