BOUNDARY VALUES OF GENERALIZED SOLUTIONS OF A HOMOGENEOUS STURM–LIOUVILLE EQUATION IN A SPACE OF VECTOR FUNCTIONS

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We consider a differential equation of the form \(-y'' + A^2y = 0\), where \(A\) is a self-adjoint operator in a Hilbert space \(H\). We show that each generalized solution of this equation in \(W_m(0, b) (0 < b < \infty, m > 0)\) has boundary values in the space \(H_{-m/2}\), where \(H_{j} (-\infty < j < \infty)\) is the Hilbert scale of spaces generated by the operator \(A\), and \(W_m(0, b)\) is the space of continuous linear functionals on \(W_m(0, b)\), the completion of the space of infinitely differentiable vector functions with compact support with respect to the norm \(\|u\|_{W_m(0, b)} = \left(\int_0^b \|u(t)\|^2_H dt\right)^{1/2}\).

It follows that each function \(u(t, x)\) which is harmonic in the strip \(G = [0, b] \times (-\infty, \infty)\) and which is in the space that is dual to \(W_2^m(\mathbb{R})\) has limiting values as \(t \to 0\) and \(t \to b\) in the space \(W_2^{-m/2}(-\infty, \infty)\).

1. Let \(H\) be a separable Hilbert space with a scalar product \((\cdot, \cdot)\) and norm \(\|\cdot\|\), and let \(L_2(H, (0, b))\) be the space of all vector functions \(u(t) (t \in [0, b])\) with values in \(H\) such that \(\int_0^b \|u(t)\|^2_H dt < \infty\).

The scalar product in \(L_2(H, (0, b))\) is defined by the formula

\[
(u, v)_{L_2(H, (0, b))} = \int_0^b (u(t), v(t)) dt.
\]

We consider the differential equation \(l[y] = 0\), where

\[
l[y] = -y'' + A^2y,
\]

and \(A\) is a self-adjoint operator in \(H\) (we can assume that \(A^2 \geq E\)).

We let \(H_j (-\infty < j < \infty)\) denote the Hilbert scale of spaces generated by the operator \(A\) (see [1] and [2]). As in [1], for integers \(m > 0\) we introduce the spaces

\[
W_m(0, b) = \{u: u \in L_2(H_m, (0, b)), u^{(m)} \in L_2(H, (0, b))\}
\]

(here \(u^{(m)}\) is the \(m\)-th derivative in the sense of the theory of distributions). By the intermediate derivative theorem (see [1]) we have \(u^{(j)} \in L_2(H_{m-j}, (0, b)) (0 < j < m)\). When equipped with the norm

\[
\|u\|_{W_m(0, b)} = \left(\int_0^b \|u(t)\|^2_H dt + \sum_{j=0}^{m-1} \|u^{(j)}(t)\|^2_{H_{m-j}(0, b)}\right)^{1/2},
\]

\(W_m(0, b)\) is a Hilbert space.

We let $\tilde{W}_m(0, b)$ denote the closure in $W_m(0, b)$ of the set $C^\infty_0(H, (0, b))$ of vector functions of the form $\tilde{\varphi} = \sum_{k=1}^{n} \varphi_k(t)f_k$, where $\varphi_k(t) \in C^\infty_0(0, b)$, the set of all scalar infinitely differentiable functions on $(0, b)$ with compact support, and $f_k \in H = \sum_{j=0}^{\infty} \mathcal{D}(A)$ ($\mathcal{D}(A)$ is the domain of definition of the operator $A$).

Since $\tilde{W}_m(0, b) \subset L_2(H, (0, b))$, where $\|u\|_{W_m(0, b)} \geq \|u\|_{L_2(H, (0, b))}$ ($u \in \tilde{W}_m(0, b)$), we can regard $\tilde{W}_m(0, b)$ as a space with a positive norm with respect to $L_2(H, (0, b))$. We let $W_m(0, b)$ denote the corresponding space with a negative norm (see [2]). Thus we have the chain of inclusions

$$ W_m(0, b) \subset L_2(H, (0, b)) \subset W_m(0, b). $$

We note that if $y \in W_m(0, b)$ and $u \in \tilde{W}_m(0, b)$, then the expression $(y, u)_{L_2(H, (0, b))}$ makes sense, denoting the action of the functional $y$ on the element $u$.

**Lemma 1.** Each element $y \in W_m(0, b)$ can be written in the form $y = y_0^m$, where $y_0 \in L_2(H, (0, b))$ (the derivative is understood in the sense of the theory of distributions).

The proof follows the usual pattern. By means of the mapping

$$ u \to \{u^{(m)}|_{m=0}\}, $$

we identify the space $W_m(0, b)$ with a closed subspace of $L_2(H_m, (0, b)) \oplus L_2(H_{m-1}, (0, b)) \oplus \ldots \oplus L_2(H, (0, b))$, and then by the Hahn–Banach and Riesz theorems each continuous linear functional $u \to Lu$ on $W_m(0, b)$ can be written in the form

$$ Lu = \sum_{j \leq m} \int_0^b (g_j(t), u^{(j)}(t))_{H_{m-j}} dt, \quad g_j \in L_2(H_{m-j}, (0, b)). $$

If $y \in W_m(0, b)$, then $y$ can be written in a form such that $y$ is uniquely determined by its values $(y, \varphi)_{L_2(H, (0, b))}$ on vector functions $\varphi \in C^\infty_0(H, (0, b))$. But

$$ (y, \varphi)_{L_2(H, (0, b))} = \sum_{j \leq m} \int_0^b (g_j(t), \varphi^{(j)}(t))_{H_{m-j}} dt = \sum_{j=m}^{\infty} \int_0^b (A^{m-j}y_j(t), \varphi^{(j)}(t))_{H_{m-j}} dt = \int_0^b (y_0(t), \varphi^{(m)}(t))_{H_{m-j}} dt, $$

where

$$ y_0(t) = \sum_{j=m}^{\infty} A^{m-j} \int_0^b \cdots \int_0^b g_j(t, \zeta_1) dt dt \cdots dt. $$

The lemma is proved.

2. As Lemmas 1 and 2 in [3] showed, a vector function $e^{-At}f$ belongs to the space $L_2(H, (0, b))$ if and only if $f \in H_{-1/2}$. This fact is generalized by the following result.

**Lemma 2.** A vector function $e^{-At}f$ ($f \in H_{p}, p \geq 0$) belongs to $W_m(0, b)$ ($m \geq 0$) if and only if $f \in H_{-(m + 1/2)}$.

**Proof.** First we show that the vector function $t^m e^{-At}f$ ($f \in H_{p}$) belongs to $W_m(0, b)$ if and only if $f \in H_{-(m + 1/2)}$.

For $f \in H_{-\frac{1}{2}}$ we have

$$ \left\|t^m e^{-At}f\right\|_{W_m(0, b)} = \int_0^b \left\|t^m e^{-At}f\right\|_{H_m} dt + \int_0^b \left\|(t^m e^{-At}f)^{(m)}\right\|_{H_m} dt = \int_0^b \left\|t^m A^m e^{-At}f\right\|^2 + \sum_{k=0}^{m} c_k t^k A^k e^{-At}f \right\| dt, $$

where the $c_k$ are constants. Therefore

$$ \left\|t^m e^{-At}f\right\|_{W_m(0, b)} = \int_0^b \left\{t^m A^m e^{-At}f + \left(\sum_{k=0}^{m} c_k t^k A^k e^{-At}f\right)^2\right\} d(E_{At}, f) = $$