A Single Groupoid Identity for Steiner Loops

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Abstract

A loop which satisfies the identities \( x^2 = e, \ x e = e x = x, \) and \( x (y x) = (x y) x = y \) is called a generalized Steiner loop. In this paper it is shown that a generalized Steiner loop is a groupoid with a single law \( x (((y y) z) x) = z. \)

1. Introduction

In [1], the author has defined the concept of a generalized triple system as follows. Let \( S \) be a set of \( v \) elements. Let \( T \) be a collection of \( b \) subsets of \( S \), each of which contains three elements arranged cyclically, and such that any ordered pair of elements of \( S \) appears in exactly one cyclic triplet (note the cyclic triplet \{a, b, c\} contains the ordered pairs \( ab, bc, ca, \) but not \( ba, cb, ac\)). When such a configuration exists we will refer to it as a generalized triple system. If we ignore the cyclic order of the triples a generalized triple system is a B.I.B.D. with parameters, \( v, k=3, \lambda=2, b=v(v-1)/3, r=v-1. \) In [1] the following results are established:

(a) Not every B.I.B.D. with the appropriate parameters can have its triples arranged cyclically so as to form a generalized triple system.
(b) Generalized triple systems exist for all values of \( v \) except \( v=6 \) or \( v \equiv 2 \mod 3. \) (\( v=1 \) is not considered an exception. The system exists vacuously.)
(c) There is a one to one correspondence between generalized triple systems of order \( v \) and quasigroups of order \( v \) satisfying the identities \( x^2 = x, (xy) x = x(yx) = y. \) We use the term generalized Steiner quasigroup to mean a quasigroup which satisfies the above identities.

2. Steiner Loops

Let \( \mathcal{G} \) be a generalized Steiner quasigroup of order \( v \) and binary operator. From \( \mathcal{G} \) a loop \( \mathcal{G}^{*} \) with operator \( * \) is constructed as follows. The elements of \( \mathcal{G}^{*} \) are the same as those of \( \mathcal{G} \) together with an extra element \( e. \) Multiplication in \( \mathcal{G}^{*} \) is defined as follows:

\[
a * e = e * a = a; \quad a * a = e \quad \text{and if} \quad a, b \in \mathcal{G} \quad \text{with} \quad a \neq b \quad \text{then} \quad a * b = a * b.
\]

It follows easily that \( \mathcal{G}^{*} \) is a loop with identities \( x * e = e * x = x, x * x = e, x * (y * x) = (x * y) * x = y. \) Also, the correspondence between generalized Steiner quasigroups and generalized Steiner loops is a bijection.

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THEOREM. A groupoid is a generalized Steiner loop if and only if it satisfies the identity $A(((BB) C) A) = C$.

Proof. It is clear that a generalized Steiner loop satisfies this identity. Conversely let $\mathcal{G}$ be a groupoid satisfying the identity

$$A(((BB) C) A) = C \quad (1)$$

for all $A, B, C$ in $\mathcal{G}$.

In (1) put $C = ((EE) F) (BB)$ and reducing obtain

$$A(FA) = ((EE) F) (BB) \quad (2)$$

From (2) $A(FA)$ is a function of $F$ only so we may put

$$A(FA) = F^* \quad (3)$$

and from (2)

$$((EE) F) (BB) = F^*. \quad (4)$$

From (1) and (3) obtain

$$((BB) C)^* = C. \quad (5)$$

In $A(FA) = F^*$ replace $A$ by $AF$ obtaining

$$(AF) (F(AF)) = F^* \quad \text{or} \quad (AF) A^* = F^*. \quad (6)$$

In (5) put $C = E(BB)$ obtaining

$$((BB) (E(BB)))^* = E(BB) \quad \text{or} \quad E(BB) = E^{**}. \quad (7)$$

In (3) put $A = GG$ obtaining $(GG) (F(GG)) = F^*$ and using (7) obtain

$$(GG) F^{**} = F^*. \quad (8)$$

In (3) put $F = A$, obtaining $A(AA) = A^*$ and using (7) obtain

$$A^{**} = A^*. \quad (9)$$

From (5) and (9) obtain

$$C = ((BB) C)^* = ((BB) C)^{**} = C^* \quad \text{or} \quad C^* = C. \quad (10)$$

From (10) the equations (3), (5), (6), (7) become

$$A(FA) = F, \quad (BB) C = C, \quad (AF) A = F, \quad E(BB) = E.$$

Now suppose $FA = GA$ then $A(FA) = A(GA)$ or $F = G$. Similarly using $(AF) A = F$ the cancellation law $AF = AG$ implies $F = G$ is obtained. From $F = (GG) F = (HH) F$