PARTIAL TRANSPOSITION AND NONEXTENDIBLE MAPS

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ABSTRACT. The wide class of nonextendible positive maps in C*-algebras [1, 3] is studied. An example is given showing that the set of nonextendible positive maps between C*-algebras \( \mathcal{A} \) and \( \mathcal{B} \) is not closed in \( L(\mathcal{A}, \mathcal{B}) \) in the weak topology.

1. PARTIAL TRANSPOSITION

Results of the theory of nonextendible positive maps [1, 3] allows us to consider a class smaller than the class of extremal positive maps. The present paper is devoted to the outline of the construction of a wide class of nonextendible positive maps which in case of mappings from \( M_2 \) into \( M_4 \) contains all examples of non-Jordanian maps known as yet. Our construction is based on the so called partial transposition (i.e. a mapping determined on a system of operators which are decomposable in the following manner: \( M = A + B \) where \( A : X_1 \rightarrow X_1 \), \( B : X_2 \rightarrow X_2 \), \( X_1 \cap X_2 \) is one-dimensional and \( A, B \) are 0 on \( X_1 \), \( X_2 \) respectively, and given by \( A + B \rightarrow A + B^T \).

For basic definitions and general theory of positive maps in C*-algebras we refer to [1-3]. We adopt here the terminology and notation of [3].

Partial ordering in any set of operators in Hilbert space is defined by \( A \geq B \) if and only if \( A - B \) is a positive operator. Hence every linear space of operators or matrices becomes a partially ordered linear space \( \mathfrak{a} \) with the cone \( \mathfrak{a}_+ \) of positive operators in \( \mathfrak{a} \). The unit of order is the identity operator or the unit of a relevant algebra of operators.

Let \( H \) be a Hilbert space, \( (\mathfrak{a}, \mathfrak{a}_+, 1) \) a partially ordered vector space with unit and

\[
\Phi: (\mathfrak{a}, \mathfrak{a}_+, 1) \rightarrow (B(H), B(H)_+, 1_H)
\]

a positive normalized map. We use the notation \( \Phi(A \otimes h) = \Phi(A)h \in H \). Every normalized positive map defines the seminorm \( \| \Phi \| \) on the algebraic tensor product \( \mathfrak{a} \otimes H \), for the strict definition of \( \| \Phi \| \) we refer to [3].

(1.1) DEFINITION. The map \( \Phi: (\mathfrak{a}, \mathfrak{a}_+, 1) \rightarrow (B(H), B(H)_+, 1_H) \) is a nonextendible positive map if and only if the set \( N_\Phi = \{ \alpha \in \mathfrak{a} \otimes H : \| \alpha \|_\Phi = 0 \} \) contains the kernel \( K \) of the mapping \( \alpha \otimes H \ni \alpha \mapsto \Phi(\alpha) \in H \) ([3] Prop. 2.2 (a), (e)).

\( \Phi \) is a quasi-nonextendible map from a subspace \( \mathfrak{a} \) of a C*-algebra into \( B(H) \) if it belongs to

Letters in Mathematical Physics 3 (1979) 319–324.
0377-9017/79/0034–0319 $00.60.
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the weak closure of the set of positive nonextendible maps in $L(a, B(H))$. If $a$ and $H$ are finite-dimensional then $\Phi$ is a pointwise limit of nonextendible maps.

Nonextendible positive maps constructed in Theorem 6.1 in [3] have the interesting property that the set $\Phi(a)$ consists of matrices of the form $A + B$, characterized by the following condition: there exist projections $P_1, P_2$ such that $E = P_1P_2 = P_2P_1$ is one-dimensional projection, $P_1 + P_2 - E = I$ and $P_1AP_1 = A$, $P_2BP_2 = B$. $P_1, P_2$ are universal for the whole $\Phi(a)$. Alternatively, there exists in the space $H$ a basis such that all elements of $\Phi(a)$ have the following form:

\[
\begin{bmatrix}
(P_1 - E)A(P_1 - E), & (P_1 - E)AE, & 0 \\
EA(P_1 - E), & E(A + B)E, & EB(P_2 - E) \\
0, & (P_2 - E)BE, & (P_2 - E)B(P_2 - E)
\end{bmatrix}.
\] (1.2)

Now let $a_1, a_2$ be two $C^*$-subalgebras of $B(H)$. Assume that $P_i \in a_i$ are units of $a_i$, $i = 1, 2$, and that $E = P_1P_2 = P_2P_1$ is one-dimensional projection belonging to $a_1 \cap a_2$. Define the following system of operators:

\[a = a_1 + a_2.\] (1.3)

Hence $a$ consists of elements of the form (1.2). The cone $M_+$ of positive elements in $a$, is the set of positive operators belonging to $a$. It follows that the embedding from $a$ into $B(H)$ (normalized if $1_H = P_1 + P_2 - E$) is a positive nonextendible map.

We have the following nice characterization of positive elements of $a$.

(1.4) PROPOSITION. Suppose that $a \subset B(H)$ is the system of operators defined by (1.3). Then $M_+ = a_1_+ + a_2_+$.

Proof. $a_1_+ + a_2_+ \subset M_+$. We have to show the inverse inclusion. Let $M \in M_+$, then $M = A + B$, with $A \in a_1$, $B \in a_2$, and $EME = zE$ where $z$ is a non-negative real number. Put

\[A(t) = A - EAE + tE, \quad B(s) = B - EBE + sE.\] (1.5)

It is enough to show that $z = z_a + z_b$ and $A(z_a), B(z_b)$ positive implies $M = A(z_a) + B(z_b) \in a_1_+ + a_2_+$. Define

\[a = \inf \{t \geq 0: A(t) \geq 0\},\] (1.6)

\[b = \inf \{s \geq 0: B(s) \geq 0\}.\] (1.7)

It is sufficient to show that $z \geq a + b$. We have

\[M = A(a) + B(b) + (z - a - b)E \geq 0.\] (1.8)