Multiplicative symmetry and related functional equations

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Summary. Let \( (G,* ) \) be a commutative monoid. Following J. G. Dhombres, we shall say that a function \( f : G \to G \) is multiplicative symmetric on \( (G, * ) \) if it satisfies the functional equation

\[
f(x * f(y)) = f(y * f(x)) \quad \text{for all } x, y \in G.
\]

 equivalently, if \( f : G \to G \) satisfies a functional equation of the following type:

\[
f(x * f(y)) = F(x, y) \quad (x, y \in G),
\]

where \( F : G \times G \to G \) is a symmetric function (possibly depending on \( f \)), then \( f \) is multiplicative symmetric on \( (G, *) \).

In Section I, we recall the results obtained for various monoids \( G \) by J. G. Dhombres and others concerning the functional equation (1) and some functional equations of the form

\[
f(x * f(y)) = F(x, y) \quad (x, y \in G),
\]

where \( F : G \times G \to G \) may depend on \( f \). We complete these results, in particular in the case where \( G \) is the field of complex numbers, and we generalize also some results by considering more general functions \( F \).

In Section II, we consider some functional equations of the form

\[
f(x * f(y)) + f(y * f(x)) = 2F(x, y) \quad (x, y \in K),
\]

where \( (K, +, \cdot ) \) is a commutative field of characteristic zero, \( * \) is either + or \( \cdot \) and \( F : K \times K \to K \) is some symmetric function which has already been considered in Section I for the functional equation (E). We investigate here the following problem: which conditions guarantee that all solutions \( f : K \to K \) of such equations are multiplicative symmetric either on \( (K, +) \) or on \( (K, \cdot ) \)? Under such conditions, these equations are equivalent to some functional equations of the form (E) for which the solutions have been given in Section I. This is a partial answer to a question asked by J. G. Dhombres in 1973.


Manuscript received July 2, 1993 and, in final form, September 29, 1994.
I. Multiplicative symmetry

A monoid \((G, \ast)\) is a set \(G\) with a binary operation \(\ast\) which is associative. Let \((G, \ast)\) be a commutative monoid.

Following J. G. Dhombres (cf. [7]), we shall say that a function \(f: G \to G\) is a multiplicative symmetric function on \((G, \ast)\) if it satisfies the functional equation

\[
f(x \ast f(y)) = f(y \ast f(x)) \quad \text{for all } x, y \in G.
\] (1)

Equivalently, if \(f: G \to G\) satisfies a functional equation of the following type:

\[
f(x \ast f(y)) = F(x, y) \quad (x, y \in G),
\]

where \(F: G \times G \to G\) is a symmetric function (possibly depending on \(f\)), then \(f\) is a multiplicative symmetric function on \((G, \ast)\).

1. The case where \((G, \ast)\) is an abelian group denoted by \((G, +)\)

In this case, J. G. Dhombres gave in [9] the general solution of the functional equation (1). He obtained also the general solution of the following functional equations, which are particular cases of (1):

\[
f(x + f(y)) = f(f(x) + f(y)) \quad (x, y \in G)
\] (2)

\[
f(x + f(y)) = f(x + y) \quad (x, y \in G)
\] (3)

\[
f(x + f(y)) = f(x) + f(y) \quad (x, y \in G).
\] (4)

We may notice that the result given in [9] for the functional equation (4) is proved for non abelian groups.

When \((G, +)\) is the additive group \((\mathbb{R}, +)\), J. G. Dhombres obtained all the continuous solutions \(f: \mathbb{R} \to \mathbb{R}\) of the functional equation (1) (cf. [9]). Namely, the only continuous solutions \(f: \mathbb{R} \to \mathbb{R}\) of the functional equation:

\[
f(x + f(y)) = f(y + f(x)) \quad (x, y \in \mathbb{R})
\] (5)

are the constant functions and \(f(x) = x + b \ (x \in \mathbb{R})\) where \(b\) is an arbitrary real number.