Multiplicative symmetry and related functional equations

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Summary. Let \((G, \ast)\) be a commutative monoid. Following J. G. Dhombres, we shall say that a function \(f: G \to G\) is multiplicative symmetric on \((G, \ast)\) if it satisfies the functional equation

\[ f(x \ast f(y)) = f(y \ast f(x)) \quad \text{for all } x, y \in G. \]  

(1)

Equivalently, if \(f: G \to G\) satisfies a functional equation of the following type:

\[ f(x \ast f(y)) = F(x, y) \quad (x, y \in G), \]

where \(F: G \times G \to G\) is a symmetric function (possibly depending on \(f\)), then \(f\) is multiplicative symmetric on \((G, \ast)\).

In Section I, we recall the results obtained for various monoids \(G\) by J. G. Dhombres and others concerning the functional equation (1) and some functional equations of the form

\[ f(x \ast f(y)) = F(x, y) \quad (x, y \in G), \]  

(\(E\))

where \(F: G \times G \to G\) may depend on \(f\). We complete these results, in particular in the case where \(G\) is the field of complex numbers, and we generalize also some results by considering more general functions \(F\).

In Section II, we consider some functional equations of the form

\[ f(x \ast f(y)) + f(y \ast f(x)) = 2F(x, y) \quad (x, y \in K), \]

where \((K, +, \cdot)\) is a commutative field of characteristic zero, \(\ast\) is either + or \(\cdot\) and \(F: K \times K \to K\) is some symmetric function which has already been considered in Section I for the functional equation (\(E\)). We investigate here the following problem: which conditions guarantee that all solutions \(f: K \to K\) of such equations are multiplicative symmetric either on \((K, +)\) or on \((K, \cdot)\)? Under such conditions, these equations are equivalent to some functional equations of the form (\(E\)) for which the solutions have been given in Section I. This is a partial answer to a question asked by J. G. Dhombres in 1973.


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I. Multiplicative symmetry

A monoid \((G, \ast)\) is a set \(G\) with a binary operation \(\ast\) which is associative. Let \((G, \ast)\) be a commutative monoid.

Following J. G. Dhombres (cf. [7]), we shall say that a function \(f: G \to G\) is a multiplicative symmetric function on \((G, \ast)\) if it satisfies the functional equation

\[
f(x \ast f(y)) = f(y \ast f(x)) \quad \text{for all } x, y \in G. \tag{1}
\]

Equivalently, if \(f: G \to G\) satisfies a functional equation of the following type:

\[
f(x \ast f(y)) = F(x, y) \quad (x, y \in G),
\]

where \(F: G \times G \to G\) is a symmetric function (possibly depending on \(f\)), then \(f\) is a multiplicative symmetric function on \((G, \ast)\).

1. The case where \((G, \ast)\) is an abelian group denoted by \((G, +)\)

In this case, J. G. Dhombres gave in [9] the general solution of the functional equation (1). He obtained also the general solution of the following functional equations, which are particular cases of (1):

\[
f(x + f(y)) = f(f(x) + f(y)) \quad (x, y \in G) \tag{2}
\]

\[
f(x + f(y)) = f(x + y) \quad (x, y \in G) \tag{3}
\]

\[
f(x + f(y)) = f(x) + f(y) \quad (x, y \in G). \tag{4}
\]

We may notice that the result given in [9] for the functional equation (4) is proved for non abelian groups.

When \((G, +)\) is the additive group \((\mathbb{R}, +)\), J. G. Dhombres obtained all the continuous solutions \(f: \mathbb{R} \to \mathbb{R}\) of the functional equation (1) (cf. [9]). Namely, the only continuous solutions \(f: \mathbb{R} \to \mathbb{R}\) of the functional equation:

\[
f(x + f(y)) = f(y + f(x)) \quad (x, y \in \mathbb{R}) \tag{5}
\]

are the constant functions and \(f(x) = x + b \ (x \in \mathbb{R})\) where \(b\) is an arbitrary real number.