Covering abelian groups with cyclic subsets

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Summary. Let $k$ and $m$ be positive integers. An abelian group $G$ is said to have an $n$-cover if there is a subset $S$ of $G$ consisting of $n$ elements such that every non-zero element of $G$ can be expressed in the form $ig$ for some element $g$ in $S$ and integer $i$, $1 \leq i \leq k$. Let $s_n(k)$ be the largest order of abelian groups that have an $n$-cover. We investigate the behavior of $s_n(k)/k$ as $k \to \infty$ and $n$ is fixed.

1. Introduction

There is a long history of tiling, packing, and covering of Euclidean space by translates of a convex body, in particular, by the cube, ball, and regular simplex. These concepts for concave bodies, even for starbodies, have been the subject of much less investigation. However, a series of papers going back to at least 1967 have been devoted to tilings and packings by translates of certain starbodies composed of cubes [9]. At first the work focussed on tilings [6]. But recently packing problems were considered [1, 5, 8]. The techniques were combinatorial, geometric, and algebraic. In this paper we examine the covering problem and some algebraic questions it suggests.

The problem of tiling $n$-space by translates of certain star bodies raised this group-theoretic question: Let $k$ and $n$ be positive integers. Does there exist a finite abelian group $G$ of order $nk + 1$ and a subset $\{g_1, g_2, \ldots, g_n\}$ of $G$ such that each non-zero element of $G$ is uniquely expressible in the form $ig_j$, $1 \leq i \leq k$, $1 \leq j \leq n$? It has been partially answered in [2, 4, 7].

If the answer is 'no', then this packing question arises: What is the order of the smallest abelian group $G$ that contains a subset $\{g_1, g_2, \ldots, g_n\}$ such that the $kn$ element $ig_j$, $1 \leq i \leq k$, $1 \leq j \leq n$ are distinct? For $n = 1, 2$ the problem is trivial. For fixed $n \geq 3$, this order is asymptotic to $2 \cos(\pi/n)k^{3/2}$ as $k \to \infty$. (See [5].)
If the answer to the first question is 'no', a covering problem also arises: What is the order of the largest abelian group that contains a subset\( S = \{g_1, g_2, \ldots, g_n\} \) such that every element in \( G - \{0\} \) is represented at least once in the form \( ig_j, 1 \leq i \leq k, 1 \leq j \leq n \)? We answer this question easily for \( n = 1 \) and \( 2 \), obtain partial results for arbitrary \( n \), make a conjecture about the behavior of this order for any fixed \( n \), and conclude by settling the case \( n = 3 \).

2. Covering non-cyclic abelian groups

For a positive integer \( k \) let \( S(k) = \{1, 2, \ldots, k\} \). Let \( G \) be an abelian group and \( S = \{g_1, g_2, \ldots, g_n\} \) a subset of \( G \) such that each element of \( G - \{0\} \) is of the form \( ig_j, i \in S(k) \) and \( g_j \in S \). In this case we call \( S \) an \( n \)-cover of \( G \) and say that \( S(k) \) \( n \)-covers \( G \).

In this section we consider coverings of non-cyclic groups. In particular we show that \( S(k) \) does not \( 3 \)-cover a non-cyclic group of order greater than \( 2k + 2 \). In Sec. 4 we show that the same result holds for cyclic groups.

The following two lemmas concern the covering of non-cyclic groups. It turns out that if a non-cyclic group has an \( n \)-cover, then its deviation from being cyclic is controlled by \( n \).

**Lemma 2.1.** Let \( S(k) \) \( n \)-cover \( G = C(r_1) \times C(r_2) \times \cdots \times C(r_u) \), where \( r_i | r_{i+1}, i = 1, 2, \ldots, u - 1 \). Then \( n \geq \frac{(r_u - 1)}{(r_1 - 1)} \).

*Proof.* If \( G \) has an \( n \)-cover so does \( G' = [C(r_1)]^u \), which is a homomorphic image of \( G \). For each \( g' \in G' \), \( |g' - \{0\}| \leq r_1 - 1 \). Thus \( G' \), hence \( G \), cannot be covered by fewer than \( (r_u - 1)/(r_1 - 1) \) elements. \( \square \)

**Lemma 2.2.** Let \( S(k) \) \( n \)-cover \( G = C(r_1) \times C(r_2) \times \cdots \times C(r_u) \), where \( r_i | r_{i+1}, i = 1, 2, \ldots, u - 1 \) and \( n \geq 2 \). Then \( r_1 r_2 \cdots r_{u-1} \leq n - 1 \).

*Proof.* Let \( \{g_1, g_2, \ldots, g_n\} \) be an \( n \)-cover of \( G \). Since \( |g_i - \{0\}| \leq r_u - 1 \), \( 1 \leq i \leq n \), we have

\[
n(r_u - 1) \geq |G| - 1,
\]

from which the lemma follows. \( \square \)

It is easy to check that, when \( n = 3 \), either lemma implies that \( G = C(r_1) \) or \( G = C(2) \times C(r_2) \) where \( r_2 \) is even. We examine the non-cyclic group \( G \), which can