On algebraic functions satisfying a class of functional equations

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1. Introduction

More than forty years ago, Mahler [4], pages 578–580, proved the following lemma as a step towards finding conditions under which certain functions take transcendental values:

If \( f(z_1, \ldots, z_n) \) is an algebraic function of the \( n \) complex variables \( z_1, \ldots, z_n \) and satisfies the functional equation

\[
 f(z_1, \ldots, z_n) = f(z_1', \ldots, z_n')
\]

where \( r > 1 \) is a rational integer, then

\[
 f(z_1, \ldots, z_n) = \eta z_1^{e_1} \cdots z_n^{e_n},
\]

where \( e_1, \ldots, e_n \) are rational numbers and \( \eta \) is an \((r - 1)\)-th root of unity.

We shall show that algebraic functions satisfying a functional equation generalising the one above must be of a very special shape. Conversely, we obtain a purely algebraic proof of the transcendence of solutions not of that shape. Our results show, for example, that the functions

\[
1 (z) = \sum_{k=0}^{\infty} z^{2^k} \quad (f_1 (z^2) = f_1 \left(\frac{z}{2}\right) - z),
\]

\[
2 (z_1, z_2) = \sum_{h=0}^{\infty} z_1^{a_h} z_2^{a_{h+1}} \quad (f_2 (z_2, z_1 z_2) = f_2 \left(\frac{z_1}{z_2}, \frac{z_2}{z_1}\right) - z_2),
\]

where the sequence \( \{a_h\} \) is given by \( a_0 = 0, a_1 = 1, a_{h+1} = a_h + a_{h-1} (h \geq 1) \), and

\[
3 (z_1, z_2) = \sum_{h=1 \atop \{k \neq 3\}}^{[k/3]} \sum_{k=-[k/3]}^{[k/3]} z_1^k z_2^k \quad (f_3 (z_1 z_2, z_1^2 z_2^2) = f_3 \left(\frac{z_1}{z_2}, \frac{z_2}{z_1}\right)),
\]

are all transcendental. Of course, analytic considerations would readily provide the same conclusions.


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This paper depends on ideas and examples mentioned by Mahler in [3] and [4]. We consider a somewhat wider class of functional equations and obtain considerably more general results.

2. Definitions and notation

Let $T = (t_{ij})$ be an $n \times n$ matrix with integer entries. To avoid degenerate cases, we assume throughout that $T$ is non-singular and that $1$ is not an eigenvalue of $T^k$ for $k = 1, 2, \ldots$

Define a transformation $T: \mathbb{C}^n \to \mathbb{C}^n$ as follows: If $z = (z_1, \ldots, z_n)$ is a point of $\mathbb{C}^n$, then $w = Tz$ is the point with coordinates

$$w_i = \prod_{j=1}^{n} z_j^{t_{ij}} \quad (1 \leq i \leq n).$$

We adopt the usual vector notation for monomials, that is, if $\mu = (\mu_1, \ldots, \mu_n)$, then we write

$$z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n} \quad (z \text{ in } \mathbb{C}^n).$$

Note that

$$(Tz)^\mu = z^\mu T \quad (z \text{ in } \mathbb{C}^n),$$

where the exponent $\mu T$ on the right is the usual product of the row vector $\mu$ and the matrix $T$.

Finally, let

$$p(z) = \sum_\mu p(\mu) z^\mu$$

be a polynomial. We define the rank of a term $p(\mu) z^\mu$, with $p(\mu) \neq 0$, to be $\mu$. Ranks are ordered lexicographically. The leading term of $p(z)$ is the term of highest rank actually occurring in $p(z)$ and the rank of $p(z)$ is the rank of its leading term.

3. The equation $f(Tz) = (af(z) + b)/(cf(z) + d)$

The main result on the functional equation given in the title of this section is the following theorem.

**THEOREM 1.** Let $a$, $b$, $c$, $d$ be constants with $ad - bc = 1$. If $f(z)$ is an algebraic function of the $n$ complex variables $z = (z_1, \ldots, z_n)$ satisfying the functional equation

$$f(Tz) = (af(z) + b)/(cf(z) + d), \quad (1)$$

then $f(z)$ is constant.